On the steady state correlation functions of open interacting systems

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Abstract. We address the existence of steady state Green-Keldysh correlation functions of interacting fermions in mesoscopic systems for both the partitioning and partition-free scenarios. Under some spectral assumptions on the non-interacting model and for sufficiently small interaction strength, we show that the system evolves to a NESS which does not depend on the profile of the time-dependent coupling strength/bias. For the partitioned setting we also show that the steady state is independent of the initial state of the inner sample. Closed formulae for the NESS two-point correlation functions (Green-Keldysh functions), in the form of a convergent expansion, are derived. In the partitioning approach, we show that the $0^{\rm th}$ order term in the interaction strength of the charge current leads to the Landauer-Büttiker formula, while the $1^{\rm st}$ order correction contains the mean-field (Hartree-Fock) results.

1 Introduction and motivation

The mathematical theory of quantum transport has attracted a lot of interest over the last decade and substantial progress has been gradually achieved. While the development of transport theory in condensed matter physics has been essentially geared towards computational techniques, the fundamental question of whether a given confined system—the *sample*—relaxes towards a stationary state when coupled to large (i.e., infinitely extended) reservoirs is much more delicate and requires a deeper analysis. To our knowledge, the first steps in this direction are due to Lebowitz and Spohn [LS, Sp] who proved the existence of a stationary state in the van Hove (weak coupling) limit and investigated their thermodynamic properties. Their results, based on the pioneering works of Davies [Da1, Da2, Da3] on the weak coupling limit, hold under very general conditions. However, due to the time rescaling inherent to this technique, they only offer a very coarse time resolution of transport phenomena. In [JP1, MMS] relaxation to a steady state of a N-level system coupled to fermionic or bosonic reservoirs has been obtained without rescaling, for small but finite coupling strength, and under much more stringent conditions. The steady state obtained in these works are analytic in the coupling strength, and to zeroth order they coincide with the weak coupling steady state of Lebowitz and Spohn. Unfortunately, these results do not cover the particularly interesting case of confined interacting fermions coupled to biased non-interacting fermionic reservoirs (or leads).

In the current paper we address the transport problem for interacting fermions in mesoscopic systems in two distinct situations which have been discussed in the physics literature.

(i) The *partitioning* scenario: initially, the interacting sample is isolated from the leads and each lead is in thermal equilibrium. At some later time t_0 the sample is (suddenly or adiabatically) coupled to the leads. In this case the driving forces which induce transport are of thermodynamical nature: the imbalance in the leads temperatures and chemical potentials.

(ii) The *partition-free* scenario: the interacting sample coupled to the free leads are initially in joint thermal equilibrium. At the later time t_0 a bias is imposed in each lead (again suddenly or adiabatically). In this case, the driving forces are of mechanical nature: the imbalance in the biases imposed on the leads.

In the special case of non-interacting fermions (and the related XY spin chain) in the partitioning scenario, conditions for relaxation to a steady state were obtained in [AH, AP, AJPP2, Ne]. The Landauer formula was derived from the Kubo formula in [CJM, CDNP]. The nonlinear Landauer-Büttiker formula was also derived in [AJPP2, Ne]. See also the seminal work of Caroli *et al* [CCNS] for a more physical approach.

Non-interacting systems in the partition-free scenario were first considered by Cini [Ci]. At the mathematical level, the existence of non-interacting steady currents and a nonlinear Landauer-Büttiker formula was also established in this setting [CNZ, CGZ].

The physical results of Cini and Caroli *et al.* did not include an important ingredient: the interaction between electrons. This last step was first achieved by Meir and Wingreen [MW]. They used the partitioning approach and the non-equilibrium Green-Keldysh functions [Ke] to write down a formula for the steady state current through an interacting region. Later on their results were extended to time-dependent transport [JWM]. The Keldysh formalism is nowadays the standard tool of physicists for transport calculations in the presence of electron-electron interactions both for steady state and transient regime (see e.g. [MSSL, TR]). The main reason for this is that the Keldysh-Green functions can be calculated from systematic many-body perturbative schemes. Nevertheless, the Keldysh formalism for transport does not provide any arguments on the actual existence of the steady state, especially in the interacting case where any explicit calculation includes approximations of the interaction self-energy. It should be mentioned here that recent numerical simulations using time-dependent density functional (TDDFT) methods suggest that systems with Hubbard-type interactions do not evolve towards a steady state [KSKVG]. Moreover, it was also shown [PFVA] that different approximation schemes for the interaction self-energy lead to different values of the long-time current.

Relaxation to a steady state for weakly interacting systems in the partitioning scenario with sudden coupling where first obtained in [FMU, JOP3, MCP]. The off-resonant regime was investigated in [CM]. The Green-Kubo formula was proven in an abstract setting along with the Onsager Reciprocity Relations in [JOP1, JOP2] and subsequently applied to interacting fermions [JOP3].

Our present work extends these results and treats the partitioning and partition-free scenarios on an equal footing. We also show that the adiabatic and sudden coupling procedures lead to the same results, provided some spectral condition on the non-interacting one-body Hamiltonian is satisfied.

We follow the scattering approach to the construction of non-equilibrium steady state advocated by Ruelle in [Ru1, Ru2] (see also [AJPP1] for a pedagogical exposition). Our analysis combines the Dyson expansion techniques developed in [BM, FMU, BMa, JOP3] with local decay estimates of the one-particle Hamiltonian [JK]. We obtain an explicit expression for the non-equilibrium steady state in the form of a convergent expansion in powers of the interaction strength. We show that this steady state does not depend on the way the coupling to the leads or the bias are switched on. Specializing our expansion to the Green-Keldysh correlation functions, we derive a few basic properties of the latter and relate them to the spectral measures of a Liouvillian describing the dynamics of the system in the GNS representation. We also briefly discuss the Hartree-Fock approximation and entropy production.

The paper is organized as follows: Section 2 introduces the setting and notation. Section 3 contains the formulation of our main results, while Section 4 gives their detailed proofs. Finally, in Section 5 we present our conclusions and outline a few open problems.

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2 The model

2.1 The one-particle setup

We consider a Fermi gas on a discrete structure $\mathcal{S}+\mathcal{R}$ (e.g., an electronic system in the tight-binding approximation). There, $\mathcal{S}$ is a finite set describing a confined sample and $\mathcal{R}=\mathcal{R}_1+\cdots+\mathcal{R}_m$ is a collection of infinitely extended reservoirs (or leads) which feed the sample $\mathcal{S}$ (See Fig. 1). For simplicity, we will assume that these reservoirs are identical semi-infinite one-dimensional regular lattices. However, our approach can easily be adapted to other geometries.

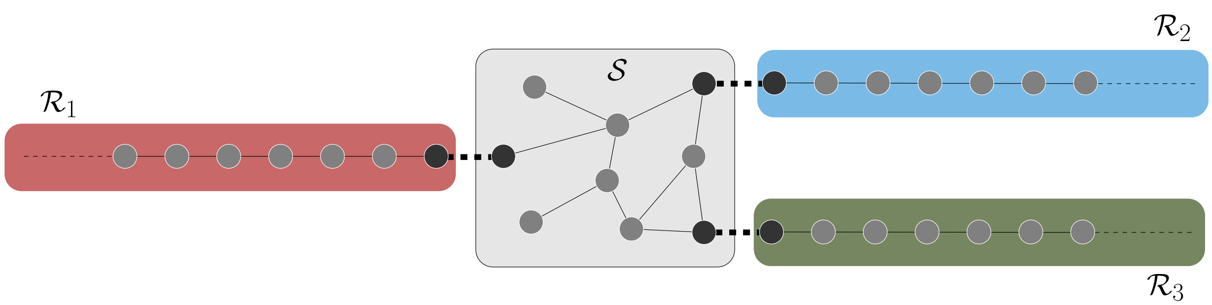

Figure 1: A finite sample $\mathcal{S}$ connected to infinite reservoirs $\mathcal{R}_1, \mathcal{R}_2, \ldots$.

The one-particle Hilbert space of the compound system is

$$\mathfrak{h}=\mathfrak{h}_{\mathcal{S}}\oplus\left(\oplus_{j=1}^m\mathfrak{h}_j\right),$$

where $\mathfrak{h}_{\mathcal{S}}=\ell^2(\mathcal{S})$ and $\mathfrak{h}_j=\ell^2(\mathbb{N})$. Let $h_{\mathcal{S}}$, a self-adjoint operator on $\mathfrak{h}_{\mathcal{S}}$, be the one-particle Hamiltonian of the isolated sample. Denote by h_j the discrete Dirichlet Laplacian on $\mathbb{N}$ with hopping constant $c_{\mathcal{R}}>0$,

$$(h_j\psi)(x)=\begin{cases}-c_{\mathcal{R}}\,\psi(1), & \text{for } x=0,\\ -c_{\mathcal{R}}\left(\psi(x-1)+\psi(x+1)\right), & \text{for } x>0.\end{cases}$$

The one-particle Hamiltonian of the reservoirs is

$$h_{\mathcal{R}}=\oplus_{j=1}^m h_j,$$

and that of the decoupled system is

$$h_{\rm D} = h_{\mathcal{S}} \oplus h_{\mathcal{R}}.$$

The coupling of the sample to the reservoirs is achieved by the tunneling Hamiltonian

$$h_{\rm T} = \sum_{j=1}^{m} d_j \left(|\delta_{0_j}\rangle\langle\phi_j| + |\phi_j\rangle\langle\delta_{0_j}| \right), \tag{2.1}$$

where $\delta_{0_j} \in \mathfrak{h}_j$ denotes the Kronecker delta at site 0 in $\mathcal{R}_j$, $\phi_j \in \mathfrak{h}_{\mathcal{S}}$ is a unit vector and $d_j \in \mathbb{R}$ a coupling constant. The one-particle Hamiltonian of the fully coupled system is

$$h_{\mathbf{0}} = h_{\rm D} + h_{\rm T}.$$

To impose biases to the leads, the one-particle Hamiltonian of the reservoirs and of the decoupled and coupled system will be changed to

$$h_{\mathcal{R},\boldsymbol{v}} = h_{\mathcal{R}} + \left(\oplus_{j=1}^{m} v_j 1_j \right), \qquad h_{{\rm D},\boldsymbol{v}} = h_{\rm D} + \left(\oplus_{j=1}^{m} v_j 1_j \right), \qquad h_{\boldsymbol{v}} = h_{{\rm D},\boldsymbol{v}} + h_{\rm T},$$

where $v_j \in \mathbb{R}$ is the bias imposed on lead $\mathcal{R}_j$, $\boldsymbol{v} = (v_1, \ldots, v_m)$, and 1_j denotes the identity on $\mathfrak{h}_j$. In the following, we will identify 1_j with the corresponding orthogonal projection acting in full one-particle Hilbert space $\mathfrak{h}$. The same convention applies to the identity $1_{\mathcal{S}/\mathcal{R}}$ on the Hilbert space $\mathfrak{h}_{\mathcal{S}/\mathcal{R}}$.

2.2 The many-body setup

We shall now describe the Fermi gas associated to the one-particle model introduced previously and extend this model by adding many-body interactions between the particles in the sample $\mathcal{S}$. In order to fix our notation and make contact with that used in the physics literature let us recall some basic facts. We refer to [BR2] for details on the algebraic framework of quantum statistical mechanics that we use here.

$\Gamma_-(\mathfrak{h})$ denotes the fermionic Fock space over $\mathfrak{h}$ and $\Gamma_-^{(n)}(\mathfrak{h}) = \mathfrak{h}^{\wedge n}$, the n-fold antisymmetric tensor power of $\mathfrak{h}$, is the n-particle sector of $\Gamma_-(\mathfrak{h})$. For $f \in \mathfrak{h}$, let $a(f)/a^*(f)$ be the annihilation/creation operator on $\Gamma_-(\mathfrak{h})$. In the following $a^{\#}$ stands for either a or a^*. The map $f \mapsto a^*(f)$ is linear while $f \mapsto a(f)$ is anti-linear, both maps being continuous, $\|a^{\#}(f)\| = \|f\|$. The underlying algebraic structure is charaterized by the canonical anticommutation relations

$$\{a(f), a^*(g)\} = \langle f|g\rangle, \qquad \{a(f), a(g)\} = 0,$$

and we denote by $\mathrm{CAR}(\mathfrak{h})$ the C^*-algebra generated by $\{a^{\#}(f) \,|\, f \in \mathfrak{h}\}$, i.e., the norm closure of the set of polynomials in the operarors $a^{\#}(f)$. Note that if $\mathfrak{g} \subset \mathfrak{h}$ is a subspace, then we can identify $\mathrm{CAR}(\mathfrak{g})$ with a subalgebra of $\mathrm{CAR}(\mathfrak{h})$.

The second quantization of a unitary operator u on $\mathfrak{h}$ is the unitary $\Gamma(u)$ on $\Gamma_-(\mathfrak{h})$ acting as $u \otimes u \otimes \cdots \otimes u$ on $\Gamma_-^{(n)}(\mathfrak{h})$. The second quantization of a self-adjoint operator q on $\mathfrak{h}$ is the self-adjoint generator $\mathrm{d}\Gamma(q)$ of the strongly continuous unitary group $\Gamma(\mathrm{e}^{\mathrm{i}tq})$, i.e., $\Gamma(\mathrm{e}^{\mathrm{i}tq}) = \mathrm{e}^{\mathrm{i}t\mathrm{d}\Gamma(q)}$. If $\{f_\iota\}_{\iota \in I}$ is an orthonormal basis of $\mathfrak{h}$ and q a bounded self-adjoint operator, then

$$\mathrm{d}\Gamma(q) = \sum_{\iota,\iota' \in I} \langle f_\iota | q f_{\iota'}\rangle a^*(f_\iota) a(f_{\iota'}),$$

holds on $\Gamma_-(\mathfrak{h})$. In particular, if q is trace class, then $\mathrm{d}\Gamma(q) \in \mathrm{CAR}(\mathfrak{h})$.

A unitary operator u on $\mathfrak{h}$ induces a Bogoliubov automorphism of $\mathrm{CAR}(\mathfrak{h})$

$$A \mapsto \gamma_u(A) = \Gamma(u) A \Gamma(u)^*,$$

such that $\gamma_u(a^\#(f)) = a^\#(uf)$. If $t \mapsto u_t$ is a strongly continuous family of unitary operators on $\mathfrak{h}$, then $t \mapsto \gamma_{u_t}$ is a strongly continuous family of Bogoliubov automorphisms of $\mathrm{CAR}(\mathfrak{h})$. In particular, if $u_t = \mathrm{e}^{\mathrm{i}tk}$ for some self-adjoint operator k on $\mathfrak{h}$, we call γ_{u_t} the quasi-free dynamics generated by k.

The quasi-free dynamics generated by the identity 1 is the gauge group of $\mathrm{CAR}(\mathfrak{h})$ and $N = \mathrm{d}\Gamma(1)$ is the number operator on $\Gamma_-(\mathfrak{h})$,

$$\vartheta^t(a^\#(f)) = \mathrm{e}^{\mathrm{i}tN} a^\#(f)\mathrm{e}^{-\mathrm{i}tN} = a^\#(\mathrm{e}^{\mathrm{i}t} f) = \begin{cases} \mathrm{e}^{-\mathrm{i}t} a(f) & \text{for } a^\# = a; \\ \mathrm{e}^{\mathrm{i}t} a^*(f) & \text{for } a^\# = a^*. \end{cases}$$

The algebra of observables of the Fermi gas is the gauge-invariant subalgebra of $\mathrm{CAR}(\mathfrak{h})$,

$$\mathrm{CAR}_\vartheta(\mathfrak{h}) = \{A \in \mathrm{CAR}(\mathfrak{h}) \,|\, \vartheta^t(A) = A \text{ for all } t \in \mathbb{R}\}.$$

It is the C^*-algebra generated by the set of all monomials in the $a^\#$ containing an equal number of a and a^* factors. Note that the map

$$\mathfrak{p}_\vartheta(A) = \int_0^{2\pi} \vartheta^t(A) \frac{\mathrm{d}t}{2\pi},$$

is a norm 1 projection onto $\mathrm{CAR}_\vartheta(\mathfrak{h})$. Thus, as a Banach space, $\mathrm{CAR}(\mathfrak{h})$ is the direct sum of $\mathrm{CAR}_\vartheta(\mathfrak{h})$ and its complement, the range of $(\mathrm{id} - \mathfrak{p}_\vartheta)$.

2.2.1 Interacting dynamics

The quasi-free dynamics generated by $h_{\boldsymbol{v}}$ describes the sample coupled to the leads and $H_{\boldsymbol{v}} = \mathrm{d}\Gamma(h_{\boldsymbol{v}})$ is the corresponding many-body Hamiltonian

$$\tau^t_{H_{\boldsymbol{v}}}(a^\#(f)) = \mathrm{e}^{\mathrm{i}tH_{\boldsymbol{v}}} a^\#(f)\mathrm{e}^{-\mathrm{i}tH_{\boldsymbol{v}}} = a^\#(\mathrm{e}^{\mathrm{i}th_{\boldsymbol{v}}} f).$$

The group $\tau_{H_{\boldsymbol{v}}}$ commutes with the gauge group ϑ so that it leaves $\mathrm{CAR}_\vartheta(\mathfrak{h})$ invariant. In the following, we shall consistently denote one-particle operators with lower-case letters and capitalize the corresponding second quantized operator, e.g., $H_{\mathcal{S}} = \mathrm{d}\Gamma(h_{\mathcal{S}})$, $H_{\mathcal{R}} = \mathrm{d}\Gamma(h_{\mathcal{R}})$, etc. We shall also denote the corresponding groups of automorphism by $\tau_{H_{\mathcal{S}}}$, $\tau_{H_{\mathcal{R}}}$, etc.

We allow for interactions between particles in the sample $\mathcal{S}$. However, particles in the leads remain free. The interaction energy within the sample is described by

$$W = \sum_{k\geq 2} \frac{1}{k!} \sum_{x_1,\ldots,x_k\in\mathcal{S}} \Phi^{(k)}(x_1,\ldots,x_k) n_{x_1}\cdots n_{x_k},$$

where $n_x = a^*(\delta_x)a(\delta_x)$ and the k-body interaction $\Phi^{(k)}$ is a completely symmetric real valued function on $\mathcal{S}^k$ which vanishes whenever two of its arguments coincide. Note that W is a self-adjoint element of $\mathrm{CAR}_\vartheta(\mathfrak{h})$. For normalization purposes, we assume that $|\Phi^{(k)}(x_1,\ldots,x_k)| \leq 1$. A typical example is provided by the second quantization of a pair potential $w(x,y) = w(y,x)$ describing the interaction energy between two particles at sites $x, y \in \mathcal{S}$. The corresponding many-body operator is

$$W = \frac{1}{2} \sum_{x,y\in\mathcal{S}} w(x,y) n_x n_y. \tag{2.2}$$

For any self-adjoint $W \in \mathrm{CAR}_\vartheta(\mathfrak{h})$ and any value of the interaction strength $\xi \in \mathbb{R}$ the operator

$$K_{\boldsymbol{v}} = H_{\boldsymbol{v}} + \xi W,$$

is self-adjoint on the domain of $H_{\boldsymbol{v}}$. Moreover $\tau_{K_{\boldsymbol{v}}}^t(A) = \mathrm{e}^{\mathrm{i}tK_{\boldsymbol{v}}} A \mathrm{e}^{-\mathrm{i}tK_{\boldsymbol{v}}}$ defines a strongly continuous group of $*$-automorphisms of $\mathrm{CAR}(\mathfrak{h})$ leaving invariant the subalgebra $\mathrm{CAR}_\vartheta(\mathfrak{h})$. This group describes the full dynamics of the Fermi gas, including interactions. It has the following norm convergent Dyson expansion

$$\tau_{K_{\boldsymbol{v}}}^t(A) = \tau_{H_{\boldsymbol{v}}}^t(A) + \sum_{n=1}^{\infty} (\mathrm{i}\xi)^n \int\limits_{0\le s_1\le\cdots\le s_n\le t} [\tau_{H_{\boldsymbol{v}}}^{s_1}(W), [\tau_{H_{\boldsymbol{v}}}^{s_2}(W), [\cdots, [\tau_{H_{\boldsymbol{v}}}^{s_n}(W), \tau_{H_{\boldsymbol{v}}}^t(A)]\cdots]]] \mathrm{d}s_1\cdots\mathrm{d}s_n.$$

2.2.2 States of the Fermi gas

A state on $\mathrm{CAR}(\mathfrak{h})$ is a linear functional

$$\begin{array}{ccc} \mathrm{CAR}(\mathfrak{h}) & \to & \mathbb{C} \\ A & \mapsto & \langle A \rangle, \end{array}$$

such that $\langle A^*A\rangle \ge 0$ for all A and $\langle I\rangle = 1$. A state is gauge-invariant if $\langle \vartheta^t(A)\rangle = \langle A\rangle$ for all $t\in\mathbb{R}$. Note that if $\langle\,\cdot\,\rangle$ is a state on $\mathrm{CAR}(\mathfrak{h})$ then its restriction to $\mathrm{CAR}_\vartheta(\mathfrak{h})$ defines a state on this subalgebra. We shall use the same notation for this restriction. Reciprocally, if $\langle\,\cdot\,\rangle$ is a state on $\mathrm{CAR}_\vartheta(\mathfrak{h})$ then $\langle \mathfrak{p}_\vartheta(\,\cdot\,)\rangle$ is a gauge-invariant state on $\mathrm{CAR}(\mathfrak{h})$.

A state $\langle\,\cdot\,\rangle$ on $\mathrm{CAR}(\mathfrak{h})$ induces a GNS representation $(\mathcal{H}, \pi, \Omega)$ where $\mathcal{H}$ is a Hilbert space, π is a $*$-morphism from $\mathrm{CAR}(\mathfrak{h})$ to the bounded linear operators on $\mathcal{H}$ and $\Omega\in\mathcal{H}$ is a unit vector such that $\pi(\mathrm{CAR}(\mathfrak{h}))\Omega$ is dense in $\mathcal{H}$ and $\langle A\rangle = (\Omega|\pi(A)\Omega)$ for all $A\in\mathrm{CAR}(\mathfrak{h})$. Let ρ be a density matrix on $\mathcal{H}$ (a non-negative, trace class operator with $\mathrm{tr}(\rho) = 1$). The map $A\mapsto \mathrm{tr}(\rho\pi(A))$ defines a state on $\mathrm{CAR}(\mathfrak{h})$. Such a state is said to be normal w.r.t. $\langle\,\cdot\,\rangle$. From the thermodynamical point of view $\langle\,\cdot\,\rangle$-normal states are close to $\langle\,\cdot\,\rangle$ and describe local perturbations of this state.

Given a self-adjoint operator ϱ on $\mathfrak{h}$ satisfying $0\le\varrho\le I$, the formula

$$\langle a^*(f_1)\cdots a^*(f_k)a(g_l)\cdots a(g_1)\rangle_\varrho = \delta_{kl}\det\{\langle g_j|\varrho f_i\rangle\}, \tag{2.3}$$

defines a unique gauge-invariant state on $\mathrm{CAR}(\mathfrak{h})$. This state is called the quasi-free state of density ϱ. It is uniquely determined by the two point function $\langle a^*(f)a(g)\rangle_\varrho = \langle g|\varrho f\rangle$. An alternative characterization of quasi-free states on $\mathrm{CAR}(\mathfrak{h})$ is the usual fermionic Wick theorem

$$\langle\varphi(f_1)\cdots\varphi(f_k)\rangle_\varrho = \begin{cases} 0, & \text{if } k \text{ is odd;} \\ \displaystyle\sum_{\pi\in\mathcal{P}_k} \varepsilon(\pi)\prod_{j=1}^{k/2}\langle\varphi(f_{\pi(2j-1)})\varphi(f_{\pi(2j)})\rangle_\varrho, & \text{if } k \text{ is even;} \end{cases} \tag{2.4}$$

where $\varphi(f) = 2^{-1/2}(a^*(f) + a(f))$ is the field operator, $\mathcal{P}_k$ denotes the set of pairings of k objects, i.e., permutations satisfying $\pi(2j-1) < \min(\pi(2j), \pi(2j+1))$ for $j = 1,\ldots,k/2$, and $\epsilon(\pi)$ is the signature of the permutation π.

Given a strongly continuous group τ of $*$-automorphisms of $\mathrm{CAR}(\mathfrak{h})$ commuting with the gauge group ϑ, a state $\langle\,\cdot\,\rangle$ is a thermal equilibrium state at inverse temperature β and chemical potential μ if it satisfies the (β,μ)-KMS condition w.r.t. τ, i.e., if for any $A, B\in\mathrm{CAR}(\mathfrak{h})$ the function

$$F_{A,B}(t) = \langle A\tau^t\circ\vartheta^{-\mu t}(B)\rangle,$$

has an analytic continuation to the strip $\{0 < \mathrm{Im}\,t < \beta\}$ with a bounded continuous extension to the closure of this strip satisfying

$$F_{A,B}(t+\mathrm{i}\beta) = \langle\tau^t\circ\vartheta^{-\mu t}(B)A\rangle.$$

We shall say that such a state is a (β,μ)-KMS state for τ.

Let k be a self-adjoint operator on $\mathfrak{h}$ and $K = \mathrm{d}\Gamma(k)$. For any $\beta,\mu\in\mathbb{R}$, the quasi-free dynamics τ_K generated by k has a unique (β,μ)-KMS state: the quasi-free state with density $\varrho_k^{\beta,\mu} = (I+\mathrm{e}^{\beta(k-\mu)})^{-1}$ which we shall denote by $\langle\,\cdot\,\rangle_K^{\beta,\mu}$. If $Q\in\mathrm{CAR}_\vartheta(\mathfrak{h})$ is self-adjoint then $K_Q = K+Q$ generates a strongly continuous group τ_{K_Q} of $*$-automorphisms of $\mathrm{CAR}(\mathfrak{h})$ leaving the subalgebra $\mathrm{CAR}_v(\mathfrak{h})$ invariant. It follows from Araki's perturbation theory that τ_{K_Q} also has a unique (β,μ)-KMS state denoted $\langle\,\cdot\,\rangle_{K_Q}^{\beta,\mu}$. Moreover, this state is normal w.r.t. $\langle\,\cdot\,\rangle_K^{\beta,\mu}$. In particular, the coupled non-interacting dynamics $\tau_{H_{\mathbf{0}}}$ and the coupled interacting dynamics $\tau_{K_{\mathbf{0}}}$ have unique (β,μ)-KMS states $\langle\,\cdot\,\rangle_{H_{\mathbf{0}}}^{\beta,\mu}$ and $\langle\,\cdot\,\rangle_{K_{\mathbf{0}}}^{\beta,\mu}$ which are mutually normal.

Remark 1. It is well known that for any $\beta>0$ and $\mu\in\mathbb{R}$ the KMS states $\langle\,\cdot\,\rangle_{H_{\mathbf{0}}}^{\beta,\mu}$ and $\langle\,\cdot\,\rangle_{K_{\mathbf{0}}}^{\beta,\mu}$ are thermodynamic limits of the familiar grand canonical Gibbs states associated to the restrictions of the Hamiltonian $H_{\mathbf{0}}$ and $K_{\mathbf{0}}$ to finitely extended reservoirs with appropriate boundary conditions. See [BR2] for details.

Remark 2. If the Hamiltonian $h_{\mathcal{S}}$ and the coupling functions ϕ_j are such that $\langle\delta_x|h_{\mathcal{S}}\delta_y\rangle$ and $\langle\delta_x|\phi_j\rangle$ are real for all $x,y\in\mathcal{S}$ then the C^*-dynamics $\tau_{K_{\boldsymbol{v}}}$ is time reversal invariant. More precisely, let Θ be the anti-linear involutive $*$-automorphism of $\mathrm{CAR}(\mathfrak{h})$ defined by $\Theta(a^\#(f)) = a^\#(\overline{f})$, where $\overline{\,\cdot\,}$ denotes the natural complex conjugation on the one-particle Hilbert space $\mathfrak{h} = \ell^2(\mathcal{S})\oplus(\oplus_{j=1}^m \ell^2(\mathbb{N}))$. Then one has

$$\tau_{K_{\boldsymbol{v}}}^t\circ\Theta = \Theta\circ\tau_{K_{\boldsymbol{v}}}^{-t}, \tag{2.5}$$

for all $t\in\mathbb{R}$. The same remark holds for the non-interacting dynamics $\tau_{H_{\boldsymbol{v}}}$ and for the decoupled dynamics $\tau_{\mathrm{D},\boldsymbol{v}}$. Moreover, the KMS-state $\langle\,\cdot\,\rangle_{K_{\mathbf{0}}}^{\beta,\mu}$ is time reversal invariant, i.e.,

$$\langle\Theta(A)\rangle_{K_{\mathbf{0}}}^{\beta,\mu} = \langle A^*\rangle_{K_{\mathbf{0}}}^{\beta,\mu},$$

holds for all $A\in\mathrm{CAR}(\mathfrak{h})$. In particular $\langle A\rangle_{K_{\mathbf{0}}}^{\beta,\mu}=0$ for any self-adjoint $A\in\mathrm{CAR}(\mathfrak{h})$ such that $\Theta(A)=-A$.

2.2.3 Current observables

Physical quantities of special interest are the charge and energy currents through the sample $\mathcal{S}$. To associate observables (i.e., elements of $\mathrm{CAR}_\vartheta(\mathfrak{h})$) to these quantities, note that the total charge inside lead $\mathcal{R}_j$ is described by $N_j = \mathrm{d}\Gamma(1_j)$. Even though this operator is not an observable (and has infinite expectation in a typical state like $\langle\,\cdot\,\rangle_{H_{\mathbf{0}}}^{\beta,\mu}$), its time derivative

$$\begin{aligned} J_j = -\left.\frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{e}^{\mathrm{i}tK_{\boldsymbol{v}}}N_j\mathrm{e}^{-\mathrm{i}tK_{\boldsymbol{v}}}\right|_{t=0} &= -\mathrm{i}[K_{\boldsymbol{v}},N_j] = -\mathrm{i}[\mathrm{d}\Gamma(h_{\mathrm{D},\boldsymbol{v}}+h_{\mathrm{T}})+\xi W,\mathrm{d}\Gamma(1_j)] \\ &= \mathrm{d}\Gamma(\mathrm{i}[1_j,h_{\mathrm{T}}]) = \mathrm{i}d_j\left(a^*(\delta_{0_j})a(\phi_j)-a^*(\phi_j)a(\delta_{0_j})\right), \end{aligned} \tag{2.6}$$

belongs to $\mathrm{CAR}_\vartheta(\mathfrak{h})$ and hence

$$\mathrm{e}^{\mathrm{i}tK_{\boldsymbol{v}}}N_j\mathrm{e}^{-\mathrm{i}tK_{\boldsymbol{v}}} - N_j = -\int_0^t \tau_{K_{\boldsymbol{v}}}^s(J_j)\,\mathrm{d}s,$$

is an observable describing the net charge transported into lead $\mathcal{R}_j$ during the period $[0,t]$. We shall consequently consider J_j as the charge current entering the sample from lead $\mathcal{R}_j$. Gauge invariance implies that the total charge inside the sample, which is described by the observable $N_{\mathcal{S}} = \mathrm{d}\Gamma(1_{\mathcal{S}})\in\mathrm{CAR}_\vartheta(\mathfrak{h})$, satisfies

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\tau_{K_{\boldsymbol{v}}}^t(N_{\mathcal{S}})\right|_{t=0} = \mathrm{i}[K_{\boldsymbol{v}},N_{\mathcal{S}}] = \mathrm{i}\left[K_{\boldsymbol{v}},N-\sum_{j=1}^m N_j\right] = \sum_{j=1}^m J_j.$$

In a similar way we define the energy currents

$$E_j = -\left.\frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{e}^{\mathrm{i}tK_{\boldsymbol{v}}}H_j\mathrm{e}^{-\mathrm{i}tK_{\boldsymbol{v}}}\right|_{t=0} = -\mathrm{i}[K_{\boldsymbol{v}}, H_j] = -\mathrm{i}[\mathrm{d}\Gamma(h_{\mathrm{D},\boldsymbol{v}} + h_{\mathrm{T}}) + \xi W, \mathrm{d}\Gamma(h_j)]$$
$$= \mathrm{d}\Gamma(\mathrm{i}[h_j, h_{\mathrm{T}}]) = -\mathrm{i}c_{\mathcal{R}}d_j\left(a^*(\delta_{1_j})a(\phi_j) - a^*(\phi_j)a(\delta_{1_j})\right), \tag{2.7}$$

and derive the conservation law

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\tau^t_{K_{\boldsymbol{v}}}(H_{\mathcal{S}} + \xi W + H_{\mathrm{T}})\right|_{t=0} = \mathrm{i}[K_{\boldsymbol{v}}, H_{\mathcal{S}} + \xi W + H_{\mathrm{T}}] = \mathrm{i}\left[K_{\boldsymbol{v}}, K_{\boldsymbol{v}} - \sum_{j=1}^m (H_j + v_jN_j)\right] = \sum_{j=1}^m E_j + v_jJ_j.$$

It follows that for any $\tau_{K_{\boldsymbol{v}}}$-invariant state $\langle\,\cdot\,\rangle$ one has the sum rules

$$\sum_{j=1}^m \langle J_j\rangle = 0, \qquad \sum_{j=1}^m \langle E_j + v_jJ_j\rangle = 0, \tag{2.8}$$

which express charge and energy conservation. Note that, despite their definition, the current observables do not depend on the bias $\boldsymbol{v}$.

Remark. Charge and energy transport within the system can also be characterized by the so called counting statistics (see [LL, ABGK, LLL] and the comprehensive review [EHM]). We shall not consider this option here and refer the reader to [DRM, JOPP, JOPS] for discussions and comparisons of the two approaches and to [FNBSJ, FNBJ] for a glance on the problem of full counting statistics in interacting non-markovian systems.

2.3 Non-equilibrium steady states

Two physically distinct ways of driving the combined system $\mathcal{S}+\mathcal{R}$ out of equilibrium have been used and discussed in the literature. In the first case, the *partitioning scenario*, one does not impose any bias in the reservoirs. The latter remain decoupled from the sample at early times $t < t_0$. During this period each reservoir is in thermal equilibrium, but the intensive thermodynamic parameters (temperatures and chemical potentials) of these reservoirs are distinct so that they are not in mutual equilibrium. At time $t = t_0$ one switches on the couplings to the sample and let the system evolve under the full unbiased dynamics $\tau_{K_{\mathbf{0}}}$. In the second case, the *partition-free scenario*, the combined system $\mathcal{S}+\mathcal{R}$ remains coupled at all times. For $t < t_0$ it is in a thermal equilibrium state associated to the unbiased dynamics $\tau_{K_{\mathbf{0}}}$. At time $t = t_0$ a bias $\boldsymbol{v} \neq \mathbf{0}$ is applied to the leads and the system then evolves according to the biased dynamics $\tau_{K_{\boldsymbol{v}}}$. In both cases, it is expected that as $t_0 \to -\infty$, the system will reach a steady state at time $t = 0$.

We shall adopt a unified approach which allows us to deal with these two scenarios on an equal footing and to consider mixed situations where both thermodynamical and mechanical forcing act on the sample.

We say that the gauge-invariant state $\langle\,\cdot\,\rangle$ on $\mathrm{CAR}(\mathfrak{h})$ is almost-$(\boldsymbol{\beta},\boldsymbol{\mu})$-KMS with $\boldsymbol{\beta} = (\beta_1,\ldots,\beta_m) \in \mathbb{R}^m_+$ and $\boldsymbol{\mu} = (\mu_1,\ldots,\mu_m) \in \mathbb{R}^m$ if it is normal w.r.t. the quasi-free state $\langle\,\cdot\,\rangle^{\boldsymbol{\beta},\boldsymbol{\mu}}_{\mathcal{R}}$ on $\mathrm{CAR}(\mathfrak{h})$ with density

$$\varrho^{\boldsymbol{\beta},\boldsymbol{\mu}}_{\mathcal{R}} = \left(\frac{1}{2}1_{\mathcal{S}}\right) \oplus \left(\bigoplus_{j=1}^m \frac{1}{1+\mathrm{e}^{\beta_j(h_j-\mu_j)}}\right). \tag{2.9}$$

The restriction of $\langle\,\cdot\,\rangle^{\boldsymbol{\beta},\boldsymbol{\mu}}_{\mathcal{R}}$ to $\mathrm{CAR}(\mathfrak{h}_{\mathcal{R}_j})$ is the unique (β_j,μ_j)-KMS state for τ_{H_j}. Its restriction to $\mathrm{CAR}(\mathfrak{h}_{\mathcal{S}})$ is the unique tracial state on this finite dimensional algebra. We also remark that if $\boldsymbol{\beta} = (\beta,\ldots,\beta)$ and $\boldsymbol{\mu} = (\mu,\ldots,\mu)$ then $\langle\,\cdot\,\rangle^{\boldsymbol{\beta},\boldsymbol{\mu}}_{\mathcal{R}}$ is the (β,μ)-KMS state for $\tau_{H_{\mathcal{R}}}$.

An almost-$(\boldsymbol{\beta},\boldsymbol{\mu})$-KMS state describes the situation where each reservoir $\mathcal{R}_j$ is near thermal equilibrium at inverse temperature β_j and chemical potential μ_j. There is however no restriction on the state of the sample $\mathcal{S}$ which can be an arbitrary gauge-invariant state on $\mathrm{CAR}(\mathfrak{h}_{\mathcal{S}})$. In particular, an almost-$(\boldsymbol{\beta},\boldsymbol{\mu})$-KMS state needs not be quasi-free or a product state.

We say that the gauge-invariant state $\langle\,\cdot\,\rangle_+^{\boldsymbol{\beta},\boldsymbol{\mu},\boldsymbol{v}}$ on $\mathrm{CAR}_\vartheta(\mathfrak{h})$ is *the* $(\boldsymbol{\beta},\boldsymbol{\mu},\boldsymbol{v})$-NESS of the system $\mathcal{S}+\mathcal{R}$ if

$$\langle A\rangle_+^{\boldsymbol{\beta},\boldsymbol{\mu},\boldsymbol{v}} = \lim_{t_0\to-\infty}\langle\tau_{K_{\boldsymbol{v}}}^{-t_0}(A)\rangle,$$

holds for *any* almost-$(\boldsymbol{\beta},\boldsymbol{\mu})$-KMS state $\langle\,\cdot\,\rangle$ and any $A\in\mathrm{CAR}_\vartheta(\mathfrak{h})$. Since

$$\langle\tau_{K_{\boldsymbol{v}}}^{t}(A)\rangle_+^{\boldsymbol{\beta},\boldsymbol{\mu},\boldsymbol{v}} = \lim_{t_0\to-\infty}\langle\tau_{K_{\boldsymbol{v}}}^{t-t_0}(A)\rangle = \langle A\rangle_+^{\boldsymbol{\beta},\boldsymbol{\mu},\boldsymbol{v}},$$

the $(\boldsymbol{\beta},\boldsymbol{\mu},\boldsymbol{v})$-NESS, if it exists, is invariant under the full dynamics $\tau_{K_{\boldsymbol{v}}}$. By definition, it is independent of the initial state of the system $\mathcal{S}$.

In the two following sections we explain how the partitioning and partition-free scenario fits into this general framework. We also introduce time-dependent protocols to study the effect of an adiabatic switching of the tunneling Hamiltonian H_{T} or of the bias $\boldsymbol{v}$.

2.3.1 The partitioning scenario

In this scenario, there is no bias in the leads, i.e., $\boldsymbol{v}=\boldsymbol{0}$ at any time. The initial state is an almost-$(\boldsymbol{\beta},\boldsymbol{\mu})$-KMS product state

$$\langle A_{\mathcal{S}}A_1\cdots A_m\rangle = \langle A_{\mathcal{S}}\rangle_{\mathcal{S}}\langle A_1\rangle_{H_{\mathcal{R}_1}}^{\beta_1,\mu_1}\cdots\langle A_m\rangle_{H_{\mathcal{R}_m}}^{\beta_m,\mu_m},$$

for $A_{\mathcal{S}}\in\mathrm{CAR}_\vartheta(\mathfrak{h}_{\mathcal{S}})$ and $A_j\in\mathrm{CAR}_\vartheta(\mathfrak{h}_j)$, where $\langle\,\cdot\,\rangle_{\mathcal{S}}$ is an arbitrary gauge-invariant state on $\mathrm{CAR}(\mathfrak{h}_{\mathcal{S}})$ and $\langle\,\cdot\,\rangle_{H_{\mathcal{R}_j}}^{\beta_j,\mu_j}$ is the (β_j,μ_j)-KMS state on $\mathrm{CAR}(\mathfrak{h}_j)$ for $\tau_{H_{\mathcal{R}_j}}$.

We shall also discuss the effect of an adiabatic switching of the coupling between $\mathcal{S}$ and $\mathcal{R}$. To this end, we replace the tunneling Hamiltonian H_{T} with the time dependent one $\chi(t/t_0)H_{\mathrm{T}}$ where $\chi:\mathbb{R}\to[0,1]$ is a continuous function such that $\chi(t)=1$ for $t\le 0$ and $\chi(t)=0$ for $t\ge 1$. Thus, for $t_0<0$, the time dependent Hamiltonian

$$K_{\boldsymbol{0}}(t/t_0) = H_{\mathcal{S}} + H_{\mathcal{R}} + \chi(t/t_0)H_{\mathrm{T}} + \xi W,$$

is self-adjoint on the domain of $H_{\mathcal{R}}$ and describes the switching of the coupling H_{T} during the time period $[t_0,0]$. It generates a strongly continuous two parameter family of unitary operators $U_{\boldsymbol{0},t_0}(t,s)$ on $\Gamma_-(\mathfrak{h})$ satisfying

$$\mathrm{i}\partial_t U_{\boldsymbol{0},t_0}(t,s)\Psi = K_{\boldsymbol{0}}(t/t_0)U_{\boldsymbol{0},t_0}(t,s)\Psi,\quad U_{\boldsymbol{0},t_0}(s,s)=I,$$

for Ψ in the domain of $H_{\mathcal{R}}$. One easily shows that the formula

$$\alpha_{\boldsymbol{0},t_0}^{s,t}(A) = U_{\boldsymbol{0},t_0}(t,s)^* A U_{\boldsymbol{0},t_0}(t,s),$$

defines a strongly continuous two parameter family of $*$-automorphisms of $\mathrm{CAR}(\mathfrak{h})$ leaving $\mathrm{CAR}_\vartheta(\mathfrak{h})$ invariant. $\alpha_{\boldsymbol{0},t_0}^{s,t}$ describes the non-autonomous evolution of the system from time s to time t under adiabatic coupling. It satisfies the composition rule

$$\alpha_{\boldsymbol{0},t_0}^{s,u}\circ\alpha_{\boldsymbol{0},t_0}^{u,t} = \alpha_{\boldsymbol{0},t_0}^{s,t},$$

for any $s,u,t\in\mathbb{R}$, and in particular $(\alpha_{\boldsymbol{0},t_0}^{s,t})^{-1} = \alpha_{\boldsymbol{0},t_0}^{t,s}$.

2.3.2 The partition-free scenario

In this case the bias $\boldsymbol{v}$ is non-zero and the initial state is a thermal equilibrium state for the unbiased full dynamics, i.e., $\langle\,\cdot\,\rangle_{K_{\boldsymbol{0}}}^{\beta,\mu}$ for some $\beta>0$ and $\mu\in\mathbb{R}$. Note that since $K_{\boldsymbol{0}}=H_{\mathcal{R}}+Q$ with $Q=H_{\mathcal{S}}+\xi W+H_{\mathrm{T}}\in\mathrm{CAR}_{\vartheta}(\mathfrak{h})$, this state is almost-$(\boldsymbol{\beta},\boldsymbol{\mu})$-KMS with $\boldsymbol{\beta}=(\beta,\ldots,\beta)$ and $\boldsymbol{\mu}=(\mu,\ldots,\mu)$.

We shall also consider the adiabatic switching of the bias via the time dependent Hamiltonian

$$K_{\boldsymbol{v}}(t/t_0)=H_{\mathcal{S}}+H_{\mathcal{R}}+H_{\mathrm{T}}+\chi(t/t_0)V_{\mathcal{R}}+\xi W,$$

where

$$V_{\mathcal{R}}=\mathrm{d}\Gamma(v_{\mathcal{R}})=\sum_{j=1}^{m}v_j\mathrm{d}\Gamma(1_j).$$

We denote by $U_{\boldsymbol{v},t_0}(t,s)$ the corresponding family of unitary propagators on the Fock space $\Gamma_-(\mathfrak{h})$, and define

$$\alpha_{\boldsymbol{v},t_0}^{s,t}(A)=U_{\boldsymbol{v},t_0}(t,s)^*AU_{\boldsymbol{v},t_0}(t,s).$$

2.3.3 NESS Green-Keldysh correlation functions

Let $\langle\,\cdot\,\rangle$ be the state of the system at time t_0. The so called lesser, greater, retarded and advanced Green-Keldysh correlation functions are defined as

$$\begin{aligned}
G^{<}(t,s;x,y)&=+\mathrm{i}\langle\tau_{K_{\boldsymbol{v}}}^{s-t_0}(a_y^*)\tau_{K_{\boldsymbol{v}}}^{t-t_0}(a_x)\rangle,\\
G^{>}(t,s;x,y)&=-\mathrm{i}\langle\tau_{K_{\boldsymbol{v}}}^{t-t_0}(a_x)\tau_{K_{\boldsymbol{v}}}^{s-t_0}(a_y^*)\rangle,\\
G^{\mathrm{r}}(t,s;x,y)&=+\mathrm{i}\theta(s-t)\langle\{\tau_{K_{\boldsymbol{v}}}^{s-t_0}(a_y^*),\tau_{K_{\boldsymbol{v}}}^{t-t_0}(a_x)\}\rangle,\\
G^{\mathrm{a}}(t,s;x,y)&=-\mathrm{i}\theta(t-s)\langle\{\tau_{K_{\boldsymbol{v}}}^{s-t_0}(a_y^*),\tau_{K_{\boldsymbol{v}}}^{t-t_0}(a_x)\}\rangle,
\end{aligned}$$

where we have set $a_x=a(\delta_x)$ for $x\in\mathcal{S}\cup\mathcal{R}$ and θ denotes the Heaviside step function.

A number of physically interesting quantities can be expressed in terms of these Green's functions, e.g., the charge density

$$\mathfrak{n}(x,t)=\langle\tau_{K_{\boldsymbol{v}}}^{t-t_0}(a_x^*a_x)\rangle=\mathrm{Im}\,G^{<}(t,t;x,x),$$

or the electric current out of lead $\mathcal{R}_j$,

$$\mathrm{j}_j(t)=\langle\tau_{K_{\boldsymbol{v}}}^{t-t_0}(J_j)\rangle=-2d_j\mathrm{Re}\sum_{x\in\mathcal{S}}G^{<}(t,t;0_j,x)\phi_j(x).$$

Assuming existence of the $(\boldsymbol{\beta},\boldsymbol{\mu},\boldsymbol{v})$-NESS, the limiting Green's functions $\lim_{t_0\to-\infty}G^{\square}(t,s;x,y)$ only depend on the time difference $t-s$, e.g.,

$$G_+^{<\boldsymbol{\beta},\boldsymbol{\mu},\boldsymbol{v}}(t-s;x,y)=\lim_{t_0\to-\infty}G^{<}(t,s;x,y)=\mathrm{i}\langle a_y^*\tau_{K_{\boldsymbol{v}}}^{t-s}(a_x)\rangle_+^{\boldsymbol{\beta},\boldsymbol{\mu},\boldsymbol{v}}.$$

Accordingly, the steady state density and currents are given by

$$\mathfrak{n}^+(x)=\mathrm{Im}\,G_+^{<\boldsymbol{\beta},\boldsymbol{\mu},\boldsymbol{v}}(0;x,x),\qquad \mathrm{j}_j^+=-2d_j\mathrm{Re}\sum_{x\in\mathcal{S}}G_+^{<\boldsymbol{\beta},\boldsymbol{\mu},\boldsymbol{v}}(0;0_j,x)\phi_j(x).$$

We shall denote the Fourier transforms of the NESS Green's functions by $\widehat{G}_+^{\square\boldsymbol{\beta},\boldsymbol{\mu},\boldsymbol{v}}(\omega;x,y)$ so that

$$G_+^{\square\boldsymbol{\beta},\boldsymbol{\mu},\boldsymbol{v}}(t;x,y)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\widehat{G}_+^{\square\boldsymbol{\beta},\boldsymbol{\mu},\boldsymbol{v}}(\omega;x,y)\mathrm{e}^{\mathrm{i}t\omega}\mathrm{d}\omega.$$

We note that, a priori, $\widehat{G}_{+}^{\square\boldsymbol{\beta},\boldsymbol{\mu},\boldsymbol{v}}(\,\cdot\,;x,y)$ is only defined as a distribution. In particular $\widehat{G}_{+}^{\mathrm{r/a}\,\boldsymbol{\beta},\boldsymbol{\mu},\boldsymbol{v}}(\omega;x,y)$ is the boundary value of an analytic functions on the upper/lower half-plane. Let us briefly explain how these distributions relate to spectral measures of a self-adjoint operator.

Let $(\mathcal{H}_+,\Omega_+,\pi_+)$ denote the GNS representation of $\mathrm{CAR}(\mathfrak{h})$ induced by the $(\boldsymbol{\beta},\boldsymbol{\mu},\boldsymbol{v})$-NESS. Let L_+ be the standard Liouvillian of the dynamics $\tau_{K_{\boldsymbol{v}}}$, i.e., the unique self-adjoint operator on $\mathcal{H}_+$ such that $\mathrm{e}^{\mathrm{i}tL_+}\pi_+(A)\mathrm{e}^{-\mathrm{i}tL_+}=\pi_+(\tau_{K_{\boldsymbol{v}}}^t(A))$ for all $A\in\mathrm{CAR}(\mathfrak{h})$ and $L_+\Omega_+=0$ (see, e.g., [AJPP1, Pi]). It immediately follows that

$$\begin{aligned} G_{+}^{<\boldsymbol{\beta},\boldsymbol{\mu},\boldsymbol{v}}(t;x,y)=\mathrm{i}\langle a_y^*\tau_{K_{\boldsymbol{v}}}^t(a_x)\rangle_+ &= \mathrm{i}(\Omega_+|\pi_+(a_y^*)\mathrm{e}^{\mathrm{i}tL_+}\pi_+(a_x)\mathrm{e}^{-\mathrm{i}tL_+}\Omega_+)\\ &=\mathrm{i}(\pi_+(a_y)\Omega_+|\mathrm{e}^{\mathrm{i}tL_+}\pi_+(a_x)\Omega_+), \end{aligned}$$

and we conclude that the lesser Green's function $G_{+}^{<\boldsymbol{\beta},\boldsymbol{\mu},\boldsymbol{v}}(\,\cdot\,;x,y)$ is essentially the Fourier transform of the spectral measure of L_+ for the vectors $\pi_+(a_x)\Omega_+$ and $\pi_+(a_y)\Omega_+$. A similar result holds for the greater Green's function. A simple calculation shows that the Fourier transform of the retarded/advanced Green's functions can be expressed in terms of the boundary value of the Borel transform of spectral measures of L_+. We note however that since the GNS representation of interacting Fermi systems is usually not explicitly known, these relations can hardly be exploited (see Section 3.4 for more concrete realizations).

2.3.4 NESS and scattering theory

In the absence of electron–electron interactions ($\xi=0$) the well known Landauer-Büttiker formalism applies and the steady state currents j_j^+ can be expressed in terms of scattering data (see Remark 1 in Section 3.1 and, e.g., [Im] for a physical introduction). In fact, it is possible to relate the NESS $\langle\,\cdot\,\rangle_+^{\boldsymbol{\beta},\boldsymbol{\mu},\boldsymbol{v}}$ to the Møller operator intertwining the one-particle dynamics of the decoupled system $\mathrm{e}^{-\mathrm{i}th_{\mathrm{D},\boldsymbol{v}}}$ to that of the coupled one $\mathrm{e}^{-\mathrm{i}th_{\boldsymbol{v}}}$ (see Equ. (3.4) below). Recently, several rigorous proofs of the Landauer-Büttiker formula have been obtained on the basis on this scattering approach to the construction of the NESS [AJPP2, CJM, Ne].

As advocated by Ruelle in [Ru1, Ru2], the scattering theory of groups of C^*-algebra automorphisms (the algebraic counterpart of the familiar Hilbert space scattering theory) provides a powerful tool for the analysis of weakly interacting many body systems. As far as we know, the use of algebraic scattering in this context can be traced back to the analysis of the s–d model of the Kondo effect by Hepp [He1] (see also [He2]). It was subsequently used by Robinson [Ro] to discuss return to equilibrium in quantum statistical mechanics. More recently, it was effectively applied to the construction of the NESS of partitioned interacting Fermi gases and to the study of their properties [DFG, FMU, FMSU, JOP3]. Let us briefly explain the main ideas behind this approach (we refer the reader to [AJPP1] for a detailed pedagogical exposition).

Assuming that at the initial time t_0 the system is in a state $\langle\,\cdot\,\rangle$ which is invariant under the decoupled and non-interacting dynamics $\tau_{H_{\mathrm{D},\boldsymbol{v}}}$ we can write the expectation value of an observable $A\in\mathrm{CAR}(\mathfrak{h})$ at time t as

$$\langle\tau_{K_{\boldsymbol{v}}}^{t-t_0}(A)\rangle=\langle\tau_{H_{\mathrm{D},\boldsymbol{v}}}^{t_0}\circ\tau_{K_{\boldsymbol{v}}}^{-t_0}(\tau_{K_{\boldsymbol{v}}}^{t}(A))\rangle.$$

If we further assume that for all $A\in\mathrm{CAR}(\mathfrak{h})$ the limit

$$\varsigma(A)=\lim_{t_0\to-\infty}\tau_{H_{\mathrm{D},\boldsymbol{v}}}^{t_0}\circ\tau_{K_{\boldsymbol{v}}}^{-t_0}(A), \tag{2.10}$$

exists in the norm of $\mathrm{CAR}(\mathfrak{h})$ then we obtain the following expression for the NESS

$$\langle\tau_{K_{\boldsymbol{v}}}^{t}(A)\rangle_+=\lim_{t_0\to-\infty}\langle\tau_{K_{\boldsymbol{v}}}^{t-t_0}(A)\rangle=\langle\varsigma(\tau_{K_{\boldsymbol{v}}}^{t}(A))\rangle.$$

The map ς defined by Equ. (2.10) is an isometric $*$-endomorphism of $\mathrm{CAR}(\mathfrak{h})$ which intertwines the two groups $\tau_{H_{\mathrm{D},\boldsymbol{v}}}$ and $\tau_{K_{\boldsymbol{v}}}$, i.e., $\tau_{H_{\mathrm{D},\boldsymbol{v}}}^t\circ\varsigma=\varsigma\circ\tau_{K_{\boldsymbol{v}}}^t$. Since it plays a similar rôle than the familiar Møller (or wave) operator of Hilbert space scattering theory, it is called Møller morphism.

To construct the Møller morphism ς and hence the NESS $\langle\,\cdot\,\rangle_+$ we shall invoke the usual chain rule of scattering theory and write ς as the composition of two Møller morphisms,

$$\varsigma = \gamma_{\omega_{\boldsymbol{v}}} \circ \varsigma_{\boldsymbol{v}},$$

where $\gamma_{\omega_{\boldsymbol{v}}}$ intertwines the decoupled non-interacting dynamics $\tau_{H_{\mathrm{D},\boldsymbol{v}}}$ and the coupled non-interacting dynamics $\tau_{H_{\boldsymbol{v}}}$ and $\varsigma_{\boldsymbol{v}}$ intertwines $\tau_{H_{\boldsymbol{v}}}$ with the coupled and interacting dynamics $\tau_{K_{\boldsymbol{v}}}$. Since $\gamma_{\omega_{\boldsymbol{v}}}$ does not involve the interaction W it can be constructed by a simple one-particle Hilbert space analysis. The construction of $\varsigma_{\boldsymbol{v}}$ is more delicate and requires a control of the Dyson expansion

$$\tau_{H_{\boldsymbol{v}}}^{-t}\circ\tau_{K_{\boldsymbol{v}}}^{t}(A) = A+\sum_{n=1}^{\infty}\xi^n \int\limits_{0\le s_1\le\cdots\le s_n\le t} \mathrm{i}[\tau_{H_{\boldsymbol{v}}}^{-s_n}(W),\mathrm{i}[\tau_{H_{\boldsymbol{v}}}^{-s_{n-1}}(W),\mathrm{i}[\cdots,\mathrm{i}[\tau_{H_{\boldsymbol{v}}}^{-s_1}(W),A]\cdots]]]\mathrm{d}s_1\cdots\mathrm{d}s_n. \tag{2.11}$$

uniformly in t up to $t=+\infty$. Such a control is possible thanks to the dispersive properties of the non-interacting dynamics $\tau_{H_{\boldsymbol{v}}}$. We shall rely on the results obtained in [JOP3] on the basis of the previous works [BM, Ev, BMa] (a similar analysis can be found in [FMU]).

Remark. A serious drawback of this strategy is the fact that the above mentioned uniform control of the Dyson expansion fails as soon as a bound state occurs in the coupled non-interacting dynamics, i.e., when the one-body Hamiltonian $h_{\boldsymbol{v}}$ acquires an eigenvalue. Moreover, the presently available techniques do not allow us to exploit the repulsive nature of the electron–electron interaction. These are two main reasons which restrict the analysis to weakly interacting systems (small values of $|\xi|$).

3 Results

To formulate our main assumption, let us define

$$v_{\mathcal{S}}(E,\boldsymbol{v}) = -\lim_{\epsilon\downarrow 0} 1_{\mathcal{S}}\, h_{\mathrm{T}}(h_{\mathcal{R}}+v_{\mathcal{R}}-E-\mathrm{i}\epsilon)^{-1}h_{\mathrm{T}}1_{\mathcal{S}},$$

for $E\in\mathbb{R}$, $\boldsymbol{v}=(v_1,\ldots,v_m)\in\mathbb{R}^m$ and $v_{\mathcal{R}}=\oplus_{j=1}^m v_j 1_j$ (see Equ. (4.4) below for a more explicit formula). The following condition ensures that the one-particle Hamiltonian $h_{\boldsymbol{v}}$ has neither an eigenvalue nor a real resonance.

($\mathbf{SP}_{\boldsymbol{v}}$) The matrix

$$\mathfrak{m}_{\boldsymbol{v}}(E) = (h_{\mathcal{S}}+v_{\mathcal{S}}(E,\boldsymbol{v})-E)^{-1}, \tag{3.1}$$

exists for all $E\in\mathbb{R}$.

In particular, this implies that $h_{\boldsymbol{v}}$ has purely absolutely continuous spectrum (see Lemma 4.1 below). We note that condition (SP$_{\boldsymbol{v}}$) imposes severe restrictions on the model. Indeed, since $h_{D,\boldsymbol{v}}$ is bounded, the Hamiltonian $h_{\boldsymbol{v}}$ necessarily acquires eigenvalues as the tunneling strength $\max_j|d_j|$ increases. Moreover, if the system $\mathcal{S}$ is not completely resonant with the leads, i.e., if $h_{D,\boldsymbol{v}}$ has non-empty discrete spectrum, then $h_{\boldsymbol{v}}$ will have eigenvalues at small tunneling strength.

By the Kato-Rosenblum theorem (see e.g., [RS3]), Condition (SP$_{\boldsymbol{v}}$) implies that the Møller operator

$$\omega_{\boldsymbol{v}} = \mathrm{s}-\lim_{t\to\infty}\mathrm{e}^{-\mathrm{i}th_{\mathrm{D},\boldsymbol{v}}}\mathrm{e}^{\mathrm{i}th_{\boldsymbol{v}}}, \tag{3.2}$$

exists and is complete. Since the absolutely continuous subspace of the decoupled Hamiltonian $h_{\mathrm{D},\boldsymbol{v}}$ is $\mathfrak{h}_{\mathcal{R}}$, $\omega_{\boldsymbol{v}}$ is unitary as a map from $\mathfrak{h}$ to $\mathfrak{h}_{\mathcal{R}}$. The associated Bogoliubov map $\gamma_{\omega_{\boldsymbol{v}}}$, characterized by $\gamma_{\omega_{\boldsymbol{v}}}(a^\#(f))=a^\#(\omega_{\boldsymbol{v}}f)$, is a $*$-isomorphism from $\mathrm{CAR}(\mathfrak{h})$ to $\mathrm{CAR}(\mathfrak{h}_{\mathcal{R}})$. Since this map is going to play an important role in the following, we introduce the short notation

$$A_{\boldsymbol{v}} = \gamma_{\omega_{\boldsymbol{v}}}(A).$$

We denote $\tau_{\mathcal{R},\boldsymbol{v}}$ the quasi-free dynamics on $\mathrm{CAR}(\mathfrak{h})$ generated by the Hamiltonian $h_{\mathcal{R},\boldsymbol{v}}$ and note that this dynamics has a natural restriction to $\mathrm{CAR}(\mathfrak{h}_{\mathcal{R}})$ for which we use the same notation. Let $\mathcal{D}_{\boldsymbol{v}}$ be the linear span of $\{\mathrm{e}^{\mathrm{i}th_{\boldsymbol{v}}}f\,|\,t\in\mathbb{R}, f\in\mathfrak{h}$ finitely supported$\}$, a dense subspace of $\mathfrak{h}$. We denote by $\mathcal{A}_{\boldsymbol{v}}$ the set of polynomials in $\{a^{\#}(f)\,|\,f\in\mathcal{D}_{\boldsymbol{v}}\}$. Finally, we set

$$\Delta_n = \{(s_1,\ldots,s_n)\in\mathbb{R}^n\,|\,0\le s_1\le s_2\le\cdots\le s_n\}.$$

3.1 Existence of the NESS

Our first result concerns the existence of the $(\boldsymbol{\beta},\boldsymbol{\mu},\boldsymbol{v})$-NESS. It is based on, and provides extensions of prior results in [FMU, JOP3, AJPP2, Ne].

Theorem 3.1. *Under Condition* ($\mathbf{SP}_{\boldsymbol{v}}$) *there exists a constant* $\bar{\xi}_{\boldsymbol{v}}>0$ *such that the following statements hold if* $|\xi|<\bar{\xi}_{\boldsymbol{v}}$.

(1) *The limit* (2.10) *exists in the norm of* $\mathrm{CAR}(\mathfrak{h})$ *for any* $A\in\mathrm{CAR}(\mathfrak{h})$ *and defines a* $*$*-isomorphism* ς *from* $\mathrm{CAR}(\mathfrak{h})$ *onto* $\mathrm{CAR}(\mathfrak{h}_{\mathcal{R}})$.

(2) *For any* $\boldsymbol{\beta}\in\mathbb{R}_+^m$, $\boldsymbol{\mu}\in\mathbb{R}^m$ *the* $(\boldsymbol{\beta},\boldsymbol{\mu},\boldsymbol{v})$*-NESS exists and is given by*

$$\mathrm{CAR}(\mathfrak{h})\ni A\mapsto\langle A\rangle_+^{\boldsymbol{\beta},\boldsymbol{\mu},\boldsymbol{v}}=\langle\varsigma(A)\rangle_{\mathcal{R}}^{\boldsymbol{\beta},\boldsymbol{\mu}}.$$

(3) *For* $A\in\mathcal{A}_{\boldsymbol{v}}$ *the Dyson expansion* (2.11) *converges up to* $t=+\infty$ *and provides the following convergent power series expansion of the NESS expectation*

$$\langle A\rangle_+^{\boldsymbol{\beta},\boldsymbol{\mu},\boldsymbol{v}}=\langle A_{\boldsymbol{v}}\rangle_{\mathcal{R}}^{\boldsymbol{\beta},\boldsymbol{\mu}}+\sum_{n=1}^{\infty}\xi^n\int\limits_{\Delta_n}C_n(A_{\boldsymbol{v}};s_1,\ldots,s_n)\mathrm{d}s_1\cdots\mathrm{d}s_n, \tag{3.3}$$

where

$$C_n(A_{\boldsymbol{v}};s_1,\ldots,s_n)=\left\langle\mathrm{i}[\tau_{\mathcal{R},\boldsymbol{v}}^{-s_n}(W_{\boldsymbol{v}}),\mathrm{i}[\tau_{\mathcal{R},\boldsymbol{v}}^{-s_{n-1}}(W_{\boldsymbol{v}}),\mathrm{i}[\cdots,\mathrm{i}[\tau_{\mathcal{R},\boldsymbol{v}}^{-s_1}(W_{\boldsymbol{v}}),A_{\boldsymbol{v}}]\cdots]]]\right\rangle_{\mathcal{R}}^{\boldsymbol{\beta},\boldsymbol{\mu}}\in L^1(\Delta_n).$$

Remark 1. In absence of interaction (i.e., for $\xi=0$) Equ. (3.3) reduces to

$$\langle A\rangle_{+,\xi=0}^{\boldsymbol{\beta},\boldsymbol{\mu},\boldsymbol{v}}=\langle\gamma_{\omega_{\boldsymbol{v}}}(A)\rangle_{\mathcal{R}}^{\boldsymbol{\beta},\boldsymbol{\mu}}, \tag{3.4}$$

and extends to all $A\in\mathrm{CAR}(\mathfrak{h})$ by continuity. One immediately deduces from Equ. (2.3) and (2.9) that this is the gauge-invariant quasi-free state on $\mathrm{CAR}(\mathfrak{h})$ with density

$$\varrho_+^{\boldsymbol{\beta},\boldsymbol{\mu},\boldsymbol{v}}=\omega_{\boldsymbol{v}}^*\varrho_{\mathcal{R}}^{\boldsymbol{\beta},\boldsymbol{\mu}}\omega_{\boldsymbol{v}}.$$

In this case one can drop Assumption ($\mathbf{SP}_{\boldsymbol{v}}$) and show that

$$\lim_{t\to\infty}\frac{1}{t}\int_{-t}^{0}\langle\tau_{H_{\boldsymbol{v}}}^{-s}(A)\rangle\,\mathrm{d}s=\langle\gamma_{\omega_{\boldsymbol{v}}}(A)\rangle_{\mathcal{R}}^{\boldsymbol{\beta},\boldsymbol{\mu}},$$

holds for any almost-$(\boldsymbol{\beta},\boldsymbol{\mu})$-KMS state $\langle\,\cdot\,\rangle$ and all $A\in\mathrm{CAR}(\mathfrak{h})$ provided the one-particle Hamiltonian $h_{\boldsymbol{v}}$ has empty singular continuous spectrum (note however that the time averaging is necessary as soon as $h_{\boldsymbol{v}}$ has non-empty point spectrum). This follows, e.g., from Theorem 3.2 in [AJPP2]. Applying Corollary 4.2 of [AJPP2] we get the following Landauer-Büttiker formulae for the mean currents in the NESS

$$\langle J_j\rangle_+^{\boldsymbol{\beta},\boldsymbol{\mu},\boldsymbol{v}}=\sum_{k=1}^m\int_{I_{j,k}}T_{jk}(E,\boldsymbol{v})\,[f(\beta_j(E-v_j-\mu_j))-f(\beta_k(E-v_k-\mu_k))]\,\mathrm{d}E \tag{3.5}$$

$$\langle E_j\rangle_+^{\boldsymbol{\beta},\boldsymbol{\mu},\boldsymbol{v}}=\sum_{k=1}^m\int_{I_{j,k}}T_{jk}(E,\boldsymbol{v})\,[f(\beta_j(E-v_j-\mu_j))-f(\beta_k(E-v_k-\mu_k))]\,(E-v_j)\mathrm{d}E \tag{3.6}$$

where $I_{j,k}=\mathrm{sp}(h_j+v_j)\cap\mathrm{sp}(h_k+v_k)$, $f(x)=(1+\mathrm{e}^x)^{-1}$ and

$$T_{jk}(E,\boldsymbol{v})=d_j^2d_k^2r(E-v_j)r(E-v_k)|\langle\phi_j|\mathfrak{m}_{\boldsymbol{v}}(E)\phi_k\rangle|^2,\qquad r(E)=\left[\frac{2}{\pi c_{\mathcal{R}}^2}\left(1-\left(\frac{E}{2c_{\mathcal{R}}}\right)^2\right)\right]^{1/2}.$$

is the transmission probability trough the sample from lead $\mathcal{R}_k$ to lead $\mathcal{R}_j$ at energy E for the one-particle Hamiltonian $h_{\boldsymbol{v}}$.

Remark 2. The intertwining property of the morphism $\gamma_{\omega_{\boldsymbol{v}}}$ allows us to rewrite Equ. (3.3) in term of the non-interacting NESS (3.4) as

$$\langle A\rangle_+^{\boldsymbol{\beta},\boldsymbol{\mu},\boldsymbol{v}}=\langle A\rangle_{+,\xi=0}^{\boldsymbol{\beta},\boldsymbol{\mu},\boldsymbol{v}}+\sum_{n=1}^\infty\xi^n\int_{\Delta_n}\Big\langle\mathrm{i}[\tau_{H_{\boldsymbol{v}}}^{-s_n}(W),\mathrm{i}[\tau_{H_{\boldsymbol{v}}}^{-s_{n-1}}(W),\mathrm{i}[\cdots,\mathrm{i}[\tau_{H_{\boldsymbol{v}}}^{-s_1}(W),A]\cdots]]]\Big\rangle_{+,\xi=0}^{\boldsymbol{\beta},\boldsymbol{\mu},\boldsymbol{v}}\mathrm{d}s_1\cdots\mathrm{d}s_n,$$

for $A\in\mathcal{A}_{\boldsymbol{v}}$.

Remark 3. It follows from Theorem 3.1 that for $A\in\mathcal{A}_{\boldsymbol{v}}$ the NESS expectation $\langle A\rangle_+^{\boldsymbol{\beta},\boldsymbol{\mu},\boldsymbol{v}}$ is an analytic function of the interaction strength ξ for $|\xi|<\bar{\xi}_{\boldsymbol{v}}$. For computational purposes, its Taylor expansion around $\xi=0$ can be obtained by iterating the integral equation

$$\eta_t(A)=A+\xi\int_0^t\mathrm{i}[\tau_{H_{\boldsymbol{v}}}^{-s}(W),\eta_s(A)]\mathrm{d}s,$$

setting $t=\infty$ in the resulting expression and writing

$$\langle A\rangle_+^{\boldsymbol{\beta},\boldsymbol{\mu},\boldsymbol{v}}=\langle\eta_{t=+\infty}(A)\rangle_{+,\xi=0}^{\boldsymbol{\beta},\boldsymbol{\mu},\boldsymbol{v}}.$$

Remark 4. The spectrum of the one-body Hamiltonian $h_{\mathcal{R},\boldsymbol{v}}$ acting on $\mathfrak{h}_{\mathcal{R}}$ being purely absolutely continuous, the quasi-free C^*-dynamical system $(\mathrm{CAR}(\mathfrak{h}_{\mathcal{R}}),\tau_{\mathcal{R},\boldsymbol{v}}^t,\langle\,\cdot\,\rangle_{\mathcal{R}}^{\boldsymbol{\beta},\boldsymbol{\mu}})$ is mixing (see, e.g., [JP2, Pi, AJPP1]). Since the $*$-isomorphism ς intertwines this system with the C^*-dynamical system $(\mathrm{CAR}(\mathfrak{h}),\tau_{K_{\boldsymbol{v}}}^t,\langle\,\cdot\,\rangle_+^{\boldsymbol{\beta},\boldsymbol{\mu},\boldsymbol{v}})$, it follows that the latter is also mixing, i.e.,

$$\lim_{t\to\infty}\langle A\tau_{K_{\boldsymbol{v}}}^t(B)\rangle_+^{\boldsymbol{\beta},\boldsymbol{\mu},\boldsymbol{v}}=\langle A\rangle_+^{\boldsymbol{\beta},\boldsymbol{\mu},\boldsymbol{v}}\langle B\rangle_+^{\boldsymbol{\beta},\boldsymbol{\mu},\boldsymbol{v}},$$

holds for all $A,B\in\mathrm{CAR}(\mathfrak{h})$. It also follows that if the leads are initially near a common equilibrium state, i.e., if $\boldsymbol{\beta}=(\beta,\ldots,\beta)$ and $\boldsymbol{\mu}=(\mu,\ldots,\mu)$, then the restriction of $\langle\,\cdot\,\rangle_{\mathcal{R}}^{\boldsymbol{\beta},\boldsymbol{\mu}}$ to $\mathrm{CAR}(\mathfrak{h}_{\mathcal{R}})$ is the unique (β,μ)-KMS state for the dynamics $\tau_{\mathcal{R}}$ and hence $\langle\,\cdot\,\rangle_+^{\boldsymbol{\beta},\boldsymbol{\mu},\boldsymbol{0}}=\langle\,\cdot\,\rangle_{K_{\boldsymbol{0}}}^{\beta,\mu}$ is the unique (β,μ)-KMS state on $\mathrm{CAR}(\mathfrak{h})$ for the zero-bias dynamics $\tau_{K_{\boldsymbol{0}}}$.

Remark 5. The linear response theory of the partitioned NESS $\langle\,\cdot\,\rangle_+^{\boldsymbol{\beta},\boldsymbol{\mu},\boldsymbol{0}}$ was established in [JOP3, JPP]. In particular the Green-Kubo formula

$$L_{jk}=\frac{1}{\beta}\,\partial_{\mu_k}\langle J_j\rangle_+^{\boldsymbol{\beta},\boldsymbol{\mu},\boldsymbol{0}}\Big|_{\boldsymbol{\beta}=(\beta,\ldots,\beta),\boldsymbol{\mu}=(\mu,\ldots,\mu)}=\frac{1}{\beta}\lim_{t\to\infty}\int_0^t\left[\int_0^\beta\langle\tau_{K_{\boldsymbol{0}}}^s(J_j)\tau_{K_{\boldsymbol{0}}}^{\mathrm{i}\theta}(J_k)\rangle_{K_{\boldsymbol{0}}}^{\beta,\mu}\mathrm{d}\theta\right]\mathrm{d}s, \tag{3.7}$$

holds for the charge current observable J_j of Equ. (2.6). If the system is time reversal invariant (see Remark 2, Section 2.2.2) then this can be rewritten as

$$L_{jk} = \frac{1}{2}\int_{-\infty}^{\infty} \langle \tau_{K_{\mathbf{0}}}^{s}(J_j)J_k\rangle_{K_{\mathbf{0}}}^{\beta,\mu} \mathrm{d}s.$$

The last formula further yields the Onsager reciprocity relation $L_{jk} = L_{kj}$ (see [JOP3, JPP] for details. Similar results hold for the energy currents).

Remark 6. To the best of our knowledge, the linear response theory of the partition-free NESS has not yet been studied. In particular we do not know if the Green-Kubo formula

$$\partial_{v_k}\langle J_j\rangle_+^{\boldsymbol{\beta},\boldsymbol{\mu},\boldsymbol{v}}\Big|_{\boldsymbol{\beta}=(\beta,\ldots,\beta),\boldsymbol{\mu}=(\mu,\ldots,\mu),\boldsymbol{v}=\mathbf{0}} = \lim_{t\to\infty}\int_0^t \left[\int_0^\beta \langle \tau_{K_{\mathbf{0}}}^{s}(J_j)\tau_{K_{\mathbf{0}}}^{\mathrm{i}\theta}(J_k)\rangle_{K_{\mathbf{0}}}^{\beta,\mu}\mathrm{d}\theta\right]\mathrm{d}s,$$

holds. We note however that this formula can be explicitly checked in the non-interacting case with the help of the Landauer-Büttiker formula (3.5). Moreover, it easily follows from Duhamel's formula and Lemma 4.7 in [JOP2] that the *finite time* Green-Kubo formula

$$\partial_{v_k}\langle \tau_{K_{\boldsymbol{v}}}^{t}(J_j)\rangle_{K_{\mathbf{0}}}^{\beta,\mu}\Big|_{\boldsymbol{v}=\mathbf{0}} = \int_0^t \left[\int_0^\beta \langle \tau_{K_{\mathbf{0}}}^{s}(J_j)\tau_{K_{\mathbf{0}}}^{\mathrm{i}\theta}(J_k)\rangle_{K_{\mathbf{0}}}^{\beta,\mu}\mathrm{d}\theta\right]\mathrm{d}s,$$

holds for all $t \geq 0$. If the system is time reversal invariant then Proposition 4.1 in [JOP2] and Remark 4 imply that

$$\lim_{t\to\infty}\partial_{v_k}\langle \tau_{K_{\boldsymbol{v}}}^{t}(J_j)\rangle_{K_{\mathbf{0}}}^{\beta,\mu}\Big|_{\boldsymbol{v}=\mathbf{0}} = \frac{\beta}{2}\int_{-\infty}^{\infty}\langle \tau_{K_{\mathbf{0}}}^{s}(J_j)J_k\rangle_{K_{\mathbf{0}}}^{\beta,\mu}\mathrm{d}s.$$

Thus, the Green-Kubo formula holds iff the limit $t\to\infty$ commutes with the partial derivative ∂_{v_k} at $\boldsymbol{v}=0$,

$$\lim_{t\to\infty}\partial_{v_k}\langle \tau_{K_{\boldsymbol{v}}}^{t}(J_j)\rangle_{K_{\mathbf{0}}}^{\beta,\mu}\Big|_{\boldsymbol{v}=\mathbf{0}} = \partial_{v_k}\lim_{t\to\infty}\langle \tau_{K_{\boldsymbol{v}}}^{t}(J_j)\rangle_{K_{\mathbf{0}}}^{\beta,\mu}\Big|_{\boldsymbol{v}=\mathbf{0}}.$$

3.2 Adiabatic schemes

Our second result shows that an adiabatic switching of the coupling between the sample and the leads or of the bias does not affect the NESS.

Theorem 3.2. (1) *Assume that Condition* (**SP**$_{\mathbf{0}}$) *holds and that* $|\xi| < \bar{\xi}_{\mathbf{0}}$. *Then the adiabatic evolution* $\alpha_{\mathbf{0},t_0}^{s,t}$ *satisfies*

$$\lim_{t_0\to-\infty}\langle \alpha_{\mathbf{0},t_0}^{t_0,0}(A)\rangle = \langle A\rangle_+^{\boldsymbol{\beta},\boldsymbol{\mu},\mathbf{0}},$$

for any $A\in\mathrm{CAR}(\mathfrak{h})$ *and any almost-*$(\boldsymbol{\beta},\boldsymbol{\mu})$*-KMS state* $\langle\,\cdot\,\rangle$*, i.e., adiabatic switching of the coupling produces the same NESS as instantaneous coupling.*

(2) *Assume that Condition* (**SP**$_{\boldsymbol{v}}$) *holds and that* $|\xi| < \bar{\xi}_{\boldsymbol{v}}$. *Then the adiabatic evolution* $\alpha_{\boldsymbol{v},t_0}^{s,t}$ *satisfies*

$$\lim_{t_0\to-\infty}\langle \alpha_{\boldsymbol{v},t_0}^{t_0,0}(A)\rangle = \langle A\rangle_+^{\boldsymbol{\beta},\boldsymbol{\mu},\boldsymbol{v}},$$

for any $A\in\mathrm{CAR}(\mathfrak{h})$ *and any almost-*$(\boldsymbol{\beta},\boldsymbol{\mu})$*-KMS state* $\langle\,\cdot\,\rangle$*, i.e., adiabatic switching of the bias produces the same NESS as instantaneous switching.*

Remark 1. Part (1) is an instance of the Narnhofer-Thirring adiabatic theorem [NT]. Our proof is patterned on the proof given in [NT]. Part (2) employs some of the ideas developed in [CDP]; here the problem is 'easier' due to the absence of point spectrum.

3.3 Entropy production

We denote Araki's relative entropy of two states by $\mathrm{S}(\,\cdot\,|\,\cdot\,)$ with the notational convention of [BR2]. The following result establishes the relation between the rate of divergence of this relative entropy along the flow of the dynamics $\tau_{K_{\boldsymbol{v}}}$ and the phenomenological notion of entropy production rate of the $(\boldsymbol{\beta},\boldsymbol{\mu},\boldsymbol{v})$-NESS.

Theorem 3.3. (1) *There exists a norm dense subset $\mathfrak{S}$ of the set of all almost-$(\boldsymbol{\beta},\boldsymbol{\mu})$-KMS states such that, for all $\langle\,\cdot\,\rangle\in\mathfrak{S}$ one has*

$$\lim_{t_0\to-\infty}\frac{1}{t_0}\mathrm{S}(\langle\tau_{K_{\boldsymbol{v}}}^{-t_0}(\,\cdot\,)\rangle|\langle\,\cdot\,\rangle)=-\sum_{j=1}^{m}\langle\beta_j(E_j-\mu_jJ_j)\rangle_+^{\boldsymbol{\beta},\boldsymbol{\mu},\boldsymbol{v}}\geq 0. \tag{3.8}$$

(2) *The mean entropy production rate in the partition-free NESS is given by*

$$\lim_{t_0\to-\infty}\frac{1}{t_0}\mathrm{S}(\langle\tau_{K_{\boldsymbol{v}}}^{-t_0}(\,\cdot\,)\rangle_{K_{\mathbf{0}}}^{\beta,\mu}|\langle\,\cdot\,\rangle_{K_{\mathbf{0}}}^{\beta,\mu})=\sum_{j=1}^{m}\beta v_j\langle J_j\rangle_+^{\beta,\mu,\boldsymbol{v}}\geq 0.$$

(3) *If $\boldsymbol{\beta}=(\beta,\ldots,\beta)$, $\boldsymbol{\mu}=(\mu,\ldots,\mu)$ and $\boldsymbol{v}=(v,\ldots,v)$, then the $(\boldsymbol{\beta},\boldsymbol{\mu},\boldsymbol{v})$-NESS is the unique (β,μ)-KMS state for the dynamics $\tau_{K_{\mathbf{0}}-vN_S}$ and all the steady currents vanish*

$$\langle J_j\rangle_+^{\boldsymbol{\beta},\boldsymbol{\mu},\boldsymbol{v}}=0,\qquad \langle E_j\rangle_+^{\boldsymbol{\beta},\boldsymbol{\mu},\boldsymbol{v}}=0.$$

The inequality on the right hand side of Equ. (3.8) is related to the second law of thermodynamics. Indeed, according to phenomenological thermodynamics, the quantity

$$-\sum_{j=1}^{m}\beta_j\left(\langle E_j\rangle_+^{\boldsymbol{\beta},\boldsymbol{\mu},\boldsymbol{v}}-\mu_j\langle J_j\rangle_+^{\boldsymbol{\beta},\boldsymbol{\mu},\boldsymbol{v}}\right),$$

can be identified with the mean rate of entropy production in the steady state $\langle\,\cdot\,\rangle_+^{\boldsymbol{\beta},\boldsymbol{\mu},\boldsymbol{v}}$ (see [Ru2, JP2, JP3] for more details). For non-interacting systems, the Landauer-Büttiker formulae (3.5),(3.6) yield the following expression of this entropy production rate

$$-\sum_{j=1}^{m}\langle\beta_j(E_j-\mu_jJ_j)\rangle_+^{\boldsymbol{\beta},\boldsymbol{\mu},\boldsymbol{v}}=\sum_{j,k=1}^{m}\int_{I_{j,k}}T_{jk}(E,\boldsymbol{v})\,[f(x_k(E))-f(x_j(E))]\,x_j(E)\frac{\mathrm{d}E}{2\pi},$$

where $x_j(E)=\beta_j(E-v_j-\mu_j)$. As shown in [AJPP2, Ne], the right hand side of this identity is strictly positive if there exists a pair (j,k) such that the transmission probability $T_{jk}(E,\boldsymbol{v})$ does not vanish identically and either $\beta_j\neq\beta_k$ or $v_j+\mu_j\neq v_k+\mu_k$. The analytic dependence of the NESS expectation on the interaction strength displayed by Equ. (3.3) allows us to conclude that this situation persists for sufficiently weak interactions (a fact already proved in [FMU]). Strict positivity of entropy production for weakly interacting fermions out of equilibrium is a generic property, as shown in [JP5]. In the more general context of open quantum systems it can also be proved in the limit of weak coupling to the reservoirs (more precisely in the van Hove scaling limit) which provides another perturbative approach to this important problem (see [LS, AS, DM, DRM]).

3.4 The NESS Green-Keldysh functions

In the next result we collect some important properties of the Green-Keldysh correlation functions of the $(\boldsymbol{\beta},\boldsymbol{\mu},\boldsymbol{v})$-NESS. In particular we relate these functions to the spectral measures of an *explicit* self-adjoint operator.

Since the restriction of the state $\langle\,\cdot\,\rangle_{\mathcal{R}}^{\boldsymbol{\beta},\boldsymbol{\mu}}$ to $\mathrm{CAR}(\mathfrak{h}_{\mathcal{R}})$ is quasi-free with density $\varrho_0=\varrho_{\mathcal{R}}^{\boldsymbol{\beta},\boldsymbol{\mu}}|_{\mathfrak{h}_{\mathcal{R}}}$, it induces a GNS representation $(\mathcal{H}_{\mathcal{R}},\Omega_{\mathcal{R}},\pi_{\mathcal{R}})$ of $\mathrm{CAR}(\mathfrak{h}_{\mathcal{R}})$ of Araki-Wyss type (see [AW, DeGe, AJPP1]). More precisely, one has

$$\mathcal{H}_{\mathcal{R}}=\Gamma_-(\mathfrak{h}_{\mathcal{R}})\otimes\Gamma_-(\mathfrak{h}_{\mathcal{R}}),\qquad \Omega_{\mathcal{R}}=\Omega\otimes\Omega,$$
$$\pi_{\mathcal{R}}(a(f))=a\left((I-\rho_0)^{1/2}f\right)\otimes I+\mathrm{e}^{\mathrm{i}\pi N}\otimes a^*\left(\overline{\rho_0^{1/2}f}\right),$$

where $\Omega\in\Gamma_-(\mathfrak{h}_{\mathcal{R}})$ denotes the Fock vacuum vector, $N=\mathrm{d}\Gamma(I)$ is the number operator and $\overline{\,\cdot\,}$ denotes the usual complex conjugation on $\mathfrak{h}_{\mathcal{R}}=\oplus_{j=1}^m\ell^2(\mathbb{N})$. The standard Liouvillian

$$L_{\mathcal{R},\boldsymbol{v}}=\mathrm{d}\Gamma(h_{\mathcal{R},\boldsymbol{v}})\otimes I-I\otimes\mathrm{d}\Gamma(h_{\mathcal{R},\boldsymbol{v}}),$$

is the unique self-adjoint operator on $\mathcal{H}_{\mathcal{R}}$ such that $\mathrm{e}^{\mathrm{i}tL_{\mathcal{R},\boldsymbol{v}}}\pi_{\mathcal{R}}(A)\mathrm{e}^{-\mathrm{i}tL_{\mathcal{R},\boldsymbol{v}}}=\pi_{\mathcal{R}}(\tau_{\mathcal{R},\boldsymbol{v}}^t(A))$ for all $A\in\mathrm{CAR}(\mathfrak{h}_{\mathcal{R}})$ and $L_{\mathcal{R},\boldsymbol{v}}\Omega_{\mathcal{R}}=0$. Apart for this eigenvector, the Liouvillian $L_{\mathcal{R},\boldsymbol{v}}$ has a purely absolutely continuous spectrum filling the entire real axis.

Theorem 3.4. *Assume that Condition* (SP$_{\boldsymbol{v}}$) *holds. Then the series*

$$A_x=a(\omega_{\boldsymbol{v}}\delta_x)+\sum_{n=1}^\infty\xi^n\int_{\Delta_n}\mathrm{i}[\tau_{\mathcal{R},\boldsymbol{v}}^{-s_n}(W_{\boldsymbol{v}}),\mathrm{i}[\tau_{\mathcal{R},\boldsymbol{v}}^{-s_{n-1}}(W_{\boldsymbol{v}}),\mathrm{i}[\cdots,\mathrm{i}[\tau_{\mathcal{R},\boldsymbol{v}}^{-s_1}(W_{\boldsymbol{v}}),a(\omega_{\boldsymbol{v}}\delta_x)]\cdots]]]\mathrm{d}s_1\cdots\mathrm{d}s_n,$$

is norm convergent for $|\xi|<\bar{\xi}_v$ and defines an element of $\mathrm{CAR}(\mathfrak{h}_{\mathcal{R}})$. For $x\in\mathcal{S}\cup\mathcal{R}$, set

$$\Psi_x=\pi_{\mathcal{R}}(A_x)\Omega_{\mathcal{R}},\qquad \Psi_x^*=\pi_{\mathcal{R}}(A_x)^*\Omega_{\mathcal{R}},$$

and denote by $\lambda_{\Phi,\Psi}$ the spectral measure of $L_{\mathcal{R},\boldsymbol{v}}$ for Φ and Ψ. Then, for any $x,y\in\mathcal{S}\cup\mathcal{R}$, one has:

(1) *Ψ_x and Ψ_x^* are orthogonal to $\Omega_{\mathcal{R}}$.*

(2) *The complex measure λ_{Ψ_y,Ψ_x} is absolutely continuous w.r.t. Lebesgue's measure and*

$$\widehat{G}_+^{<\boldsymbol{\beta},\boldsymbol{\mu},\boldsymbol{v}}(\omega;x,y)=2\pi\mathrm{i}\frac{\mathrm{d}\lambda_{\Psi_y,\Psi_x}(\omega)}{\mathrm{d}\omega}.$$

(3) *The complex measure $\lambda_{\Psi_x^*,\Psi_y^*}$ is absolutely continuous w.r.t. Lebesgue's measure and*

$$\widehat{G}_+^{>\boldsymbol{\beta},\boldsymbol{\mu},\boldsymbol{v}}(\omega;x,y)=-2\pi\mathrm{i}\frac{\mathrm{d}\lambda_{\Psi_x^*,\Psi_y^*}(-\omega)}{\mathrm{d}\omega}.$$

In the remaining statements, $\square$ stands for either $<$ or $>$.

(4) *$G_+^{\square\boldsymbol{\beta},\boldsymbol{\mu},\boldsymbol{v}}(t;x,y)$ extends to an entire analytic functions of t. Moreover, for all $n\geq0$ and $z\in\mathbb{C}$,*

$$\lim_{t\to\pm\infty}\partial_z^nG_+^{\square\boldsymbol{\beta},\boldsymbol{\mu},\boldsymbol{v}}(z+t;x,y)=0.$$

(5) *There exists $\theta>0$ such that*

$$\sup_{|\mathrm{Im}\,z|\leq\theta}\int_{-\infty}^\infty\left|G_+^{\square\boldsymbol{\beta},\boldsymbol{\mu},\boldsymbol{v}}(z+t;x,y)\right|\mathrm{d}t<\infty.$$

for all $x,y\in\mathcal{S}\cup\mathcal{R}$.

(6) *$\widehat{G}_+^{\square\boldsymbol{\beta},\boldsymbol{\mu},\boldsymbol{v}}(\omega;x,y)$ is a continuous function of ω. Moreover,*

$$\sup_{\omega\in\mathbb{R}}\mathrm{e}^{\theta|\omega|}\left|\widehat{G}_+^{\square\boldsymbol{\beta},\boldsymbol{\mu},\boldsymbol{v}}(\omega;x,y)\right|<\infty,$$

holds for all $x,y\in\mathcal{S}\cup\mathcal{R}$ with the same θ as in Part (5).

3.5 The Hartree-Fock approximation

In this section we focus on two-body interactions W as given by Equ. (2.2). Recall that $w(x,x)=0$ for all $x\in\mathcal{S}$ and define the Hartree and exchange interactions by

$$W_{\mathrm{H}}=\sum_{x,y\in\mathcal{S}}w(x,y)\langle a_y^*a_y\rangle_{+,\xi=0}^{\boldsymbol{\beta},\boldsymbol{\mu},\boldsymbol{v}}\,a_x^*a_x=\mathrm{d}\Gamma(v_{\mathrm{H}}),$$
$$W_{\mathrm{X}}=\sum_{x,y\in\mathcal{S}}w(x,y)\langle a_y^*a_x\rangle_{+,\xi=0}^{\boldsymbol{\beta},\boldsymbol{\mu},\boldsymbol{v}}\,a_x^*a_y=\mathrm{d}\Gamma(v_{\mathrm{X}}),$$

and the Hartree-Fock interaction $W_{\mathrm{HF}}=W_{\mathrm{H}}-W_{\mathrm{X}}=\mathrm{d}\Gamma(v_{\mathrm{HF}})$. Recall also that $\langle\,\cdot\,\rangle_{+,\xi=0}^{\boldsymbol{\beta},\boldsymbol{\mu},\boldsymbol{v}}$ denotes the non-interacting NESS given by Equ. (3.4). With the notation of Remark 1 in Section 3.1 one has for $x,y\in\mathcal{S}$,

$$\begin{aligned}\langle a_y^*a_x\rangle_{+,\xi=0}^{\boldsymbol{\beta},\boldsymbol{\mu},\boldsymbol{v}}&=\langle\delta_x|\omega_{\boldsymbol{v}}^*\varrho_{\mathcal{R}}^{\boldsymbol{\beta},\boldsymbol{\mu}}\omega_{\boldsymbol{v}}\delta_y\rangle\\&=\sum_{j=1}^m d_j^2\int_{v_j-2c_{\mathcal{R}}}^{v_j+2c_{\mathcal{R}}}f(\beta_j(E-v_j-\mu_j))r(E-v_j)\langle\delta_x|\mathfrak{m}_{\boldsymbol{v}}(E)\phi_j\rangle\langle\phi_j|\mathfrak{m}_{\boldsymbol{v}}(E)^*\delta_y\rangle\,\frac{\mathrm{d}E}{\sqrt{2\pi}}.\end{aligned}$$

The one-particle Hartree-Fock Hamiltonian $h_{\mathrm{HF},\boldsymbol{v}}=h_{\boldsymbol{v}}+\xi v_{\mathrm{HF}}$ generates a quasi-free dynamics $\tau_{H_{\mathrm{HF},\boldsymbol{v}}}$ on $\mathrm{CAR}(\mathfrak{h})$.

Our last result shows that the Green-Keldysh correlation functions of the Hartree-Fock dynamics $G_{\mathrm{HF}+}^{\square\boldsymbol{\beta},\boldsymbol{\mu},\boldsymbol{v}}$ coincide with the one of the fully interactinng $(\boldsymbol{\beta},\boldsymbol{\mu},\boldsymbol{v})$-NESS to first order in the interaction strength.

Theorem 3.5. *Assume that Condition* (**SP**$_{\boldsymbol{v}}$) *holds.*

(1) *If ξ is small enough then the limit*

$$\langle A\rangle_{\mathrm{HF}+}^{\boldsymbol{\beta},\boldsymbol{\mu},\boldsymbol{v}}=\lim_{t_0\to-\infty}\langle\tau_{H_{\mathrm{HF},\boldsymbol{v}}}^{-t_0}(A)\rangle,$$

exists for all $A\in\mathrm{CAR}(\mathfrak{h})$. Moreover this Hartree-Fock $(\boldsymbol{\beta},\boldsymbol{\mu},\boldsymbol{v})$-NESS is given by $\langle A\rangle_{\mathrm{HF}+}^{\boldsymbol{\beta},\boldsymbol{\mu},\boldsymbol{v}}=\langle\gamma_{\omega_{\mathrm{HF},\boldsymbol{v}}}(A)\rangle_{\mathcal{R}}^{\boldsymbol{\beta},\boldsymbol{\mu}}$ where

$$\omega_{\mathrm{HF},\boldsymbol{v}}=\mathrm{s}-\lim_{t\to\infty}\mathrm{e}^{-ith_{\mathrm{D},\boldsymbol{v}}}\mathrm{e}^{ith_{\mathrm{HF},\boldsymbol{v}}}.$$

(2) *Denote by $G_{\mathrm{HF}+}^{\square\boldsymbol{\beta},\boldsymbol{\mu},\boldsymbol{v}}(t;x,y)$ the Green-Keldysh correlation functions of the Hartree-Fock NESS. Then*

$$G_+^{\square}(t;x,y)=G_{\mathrm{HF}+}^{\square}(t;x,y)+\mathcal{O}(\xi^2)\tag{3.9}$$

as $\xi\to0$. Moreover, the error term is locally uniform in x, y and t.

Since the NESS expectation of the energy and charge currents can be expressed in terms of the lesser Green-Keldysh function, one has in particular

$$\langle J_j\rangle_+^{\boldsymbol{\beta},\boldsymbol{\mu},\boldsymbol{v}}=\langle J_j\rangle_{\mathrm{HF}+}^{\boldsymbol{\beta},\boldsymbol{\mu},\boldsymbol{v}}+\mathcal{O}(\xi^2),\qquad\langle E_j\rangle_+^{\boldsymbol{\beta},\boldsymbol{\mu},\boldsymbol{v}}=\langle E_j\rangle_{\mathrm{HF}+}^{\boldsymbol{\beta},\boldsymbol{\mu},\boldsymbol{v}}+\mathcal{O}(\xi^2).$$

Moreover, the Hartree-Fock steady currents are given by the Landauer-Büttiker formulae (3.5), (3.6) with the transmission probability

$$T_{jk}(E,\boldsymbol{v})=d_j^2d_k^2r(E-v_j)r(E-v_k)|\langle\phi_j|\mathfrak{m}_{\mathrm{HF},\boldsymbol{v}(E)}\phi_k\rangle|^2,$$

where $\mathfrak{m}_{\mathrm{HF},\boldsymbol{v}(E)}=(h_{\mathcal{S}}+\xi v_{\mathrm{HF}}+v_{\mathcal{S}}(E,\boldsymbol{v})-E)^{-1}$.

4 Proofs

4.1 Preliminaries

In this section we state and prove a few technical facts which will be used in the proof of our main results. We start with some notation. Recall that $1_{\mathcal{S}}$, $1_{\mathcal{R}}$ and 1_j denote the orthogonal projections on $\mathfrak{h}_{\mathcal{S}}$, $\mathfrak{h}_{\mathcal{R}}$ and $\mathfrak{h}_j$ in $\mathfrak{h}$. We set $x = \oplus_{j=1}^m x_j$ where x_j is the position operator on lead $\mathcal{R}_j$ and use the convention $\langle x\rangle = (1+|x|)$. Thus, the operator $\langle x\rangle$ acts as the identity on $\mathfrak{h}_{\mathcal{S}}$ and as $(1+x_j)$ on $\mathfrak{h}_j$. In particular, one has $h_{\mathrm{T}} = \langle x\rangle^{\sigma} h_{\mathrm{T}} = h_{\mathrm{T}}\langle x\rangle^{\sigma}$ for arbitrary $\sigma\in\mathbb{R}$.

Lemma 4.1. *If Condition* **(SP$_{\boldsymbol{v}}$)** *is satisfied then the one particle Hamiltonian $h_{\boldsymbol{v}}$ has purely absolutely continuous spectrum. Moreover, for $\sigma > 5/2$ they are constants C_σ and c_σ such that the local decay estimates*

$$\|\langle x\rangle^{-\sigma}\mathrm{e}^{\mathrm{i}(t+\mathrm{i}\theta)h_{\boldsymbol{v}}}\langle x\rangle^{-\sigma}\| \le C_\sigma \mathrm{e}^{c_\sigma|\theta|}\langle t\rangle^{-3/2}, \tag{4.1}$$

$$\|\langle x\rangle^{-\sigma}\varrho_+^{\boldsymbol{\beta},\boldsymbol{\mu},\boldsymbol{v}}\mathrm{e}^{\mathrm{i}(t+\mathrm{i}\theta)h_{\boldsymbol{v}}}\langle x\rangle^{-\sigma}\| \le C_\sigma \mathrm{e}^{c_\sigma|\theta|}\langle t\rangle^{-3/2}, \tag{4.2}$$

hold for all $t,\theta\in\mathbb{R}$.

Proof. Define

$$d_{\mathcal{R}}(z) = 1_{\mathcal{R}}\langle x\rangle^{-\sigma}(h_{\mathcal{R}}+v_{\mathcal{R}}-z)^{-1}\langle x\rangle^{-\sigma}1_{\mathcal{R}},$$

for $\operatorname{Im} z\neq 0$. The explicit formula for the resolvent of the Dirichlet Laplacian on $\mathbb{N}$ yields, for $x,y\in\mathcal{R}_j$, $\theta\in[0,\pi]$ and $\chi>0$,

$$\begin{aligned}\lim_{\epsilon\downarrow 0}\langle\delta_x|(h_j-2c_{\mathcal{R}}\cos\theta-\mathrm{i}\epsilon)^{-1}\delta_y\rangle &= -\frac{\mathrm{e}^{-\mathrm{i}\theta|x-y|}-\mathrm{e}^{-\mathrm{i}\theta(x+y+2)}}{2\mathrm{i}c_{\mathcal{R}}\sin\theta},\\ \langle\delta_x|(h_j\mp 2c_{\mathcal{R}}\cosh\chi)^{-1}\delta_y\rangle &= -\frac{\mathrm{e}^{-\chi|x-y|}-\mathrm{e}^{-\chi(x+y+2)}}{2c_{\mathcal{R}}\sinh\chi}(\pm 1)^{x+y+1}.\end{aligned} \tag{4.3}$$

For $\sigma>1/2$, it follows that the function $d_{\mathcal{R}}(z)$, taking values in the Hilbert-Schmidt operators on $\mathfrak{h}_{\mathcal{R}}$, has boundary values $d_{\mathcal{R}}(E) = \lim_{\epsilon\downarrow 0} d_{\mathcal{R}}(E+\mathrm{i}\epsilon)$ at every point $E\in\mathbb{R}$ and $v_{\mathcal{S}}(E,\boldsymbol{v}) = h_{\mathrm{T}}d_{\mathcal{R}}(E)h_{\mathrm{T}}$. Moreover, a simple calculation shows that if $\sigma>5/2$, then the following holds:

(1) $d_{\mathcal{R}}(E)$ is continuous on $\mathbb{R}$.

(2) $d_{\mathcal{R}}(E)$ is twice continuously differentiable on $\mathbb{R}\setminus\mathcal{T}$ where $\mathcal{T} = \{v_j\pm 2c_{\mathcal{R}}\,|\,j=1,\ldots,m\}$ is the set of thresholds of $h_{\mathcal{R}}+v_{\mathcal{R}}$.

(3) For $k=1,2$,

$$\partial_E^k\, d_{\mathcal{R}}(E) = \mathcal{O}(\delta(E)^{-k+1/2}),$$

as $\delta(E)\to 0$, where $\delta(E) = \operatorname{dist}(E,\mathcal{T})$.

Note that the class of operator valued functions satisfying Conditions (1)–(3) form an algebra.

The Feshbach-Schur formula and Condition **(SP$_{\boldsymbol{v}}$)** imply that for every $E\in\mathbb{R}$ the weighted resolvent of $h_{\boldsymbol{v}}$ has a boundary value $d(E) = \lim_{\epsilon\downarrow 0}\langle x\rangle^{-\sigma}(h_{\boldsymbol{v}}-E-\mathrm{i}\epsilon)^{-1}\langle x\rangle^{-\sigma}$ given by

$$d(E) = \begin{bmatrix}\mathfrak{m}_{\boldsymbol{v}}(E) & -\mathfrak{m}_{\boldsymbol{v}}(E)h_{\mathrm{T}}d_{\mathcal{R}}(E)\\ -d_{\mathcal{R}}(E)h_{\mathrm{T}}\mathfrak{m}_{\boldsymbol{v}}(E) & d_{\mathcal{R}}(E)+d_{\mathcal{R}}(E)h_{\mathrm{T}}\mathfrak{m}_{\boldsymbol{v}}(E)h_{\mathrm{T}}d_{\mathcal{R}}(E)\end{bmatrix}.$$

From these expressions, one concludes that $d(E)$ also satisfies the above properties (1)–(3). In particular, this implies that $h_{\boldsymbol{v}}$ has purely absolutely continuous spectrum.

Denote by $\Sigma_1, \Sigma_2, \ldots$ the connected components of the bounded open set $\cup_k]v_k - 2c_{\mathcal{R}}, v_k + 2c_{\mathcal{R}}[\setminus \mathcal{T}$. By Cauchy's formula one has

$$\langle x\rangle^{-\sigma} \mathrm{e}^{\mathrm{i}th_{\boldsymbol{v}}} \langle x\rangle^{-\sigma} = \frac{1}{\pi} \sum_k \int_{\Sigma_k} \mathrm{e}^{\mathrm{i}tE} \operatorname{Im} d(E) \mathrm{d}E,$$

for $t \in \mathbb{R}$ and the local decay estimate (4.1) with $\theta = 0$ follows from Lemma 10.2 in [JK].

To prove (4.2) note that $\varrho_+^{\boldsymbol{\beta},\boldsymbol{\mu},\boldsymbol{v}} \mathrm{e}^{\mathrm{i}th_{\boldsymbol{v}}} = \sum_j \omega_{\boldsymbol{v}}^* 1_j f_j(h_j) \mathrm{e}^{\mathrm{i}t(h_j+v_j)} 1_j \omega_{\boldsymbol{v}}$ where $f_j(E) = (1 + \mathrm{e}^{\beta_j(E-\mu_j)})^{-1}$. Let U_j denote the unitary map from $\mathfrak{h}_j$ to the spectral representation of $h_j + v_j$ in $L^2([v_j - 2c_{\mathcal{R}}, v_j + 2c_{\mathcal{R}}], \mathrm{d}E)$. From the stationary representation of the Møller operator (see, e.g., [P]) we deduce

$$\left(U_j 1_j (\omega_{\boldsymbol{v}} - I)\langle x\rangle^{-\sigma} g\right)(E) = -\left(U_j 1_j h_{\mathrm{T}} d(E)^* g\right)(E),$$

which implies

$$\langle x\rangle^{-\sigma} \varrho_+^{\boldsymbol{\beta},\boldsymbol{\mu},\boldsymbol{v}} \mathrm{e}^{\mathrm{i}th_{\boldsymbol{v}}} \langle x\rangle^{-\sigma} = \frac{1}{\pi} \sum_{j=1}^m \sum_k \int_{\Sigma_k} \mathrm{e}^{\mathrm{i}tE} f_j(E - v_j) \left(I - d(E) h_{\mathrm{T}}\right) 1_j \mathrm{Im}\left(d_{\mathcal{R}}(E)\right) 1_j \left(I - h_{\mathrm{T}} d(E)^*\right) \mathrm{d}E.$$

Applying again Lemma 10.2 of [JK] yields (4.2) with $\theta = 0$.

To extend our estimates to non-zero θ, we write

$$\mathrm{e}^{\mathrm{i}(t+\mathrm{i}\theta)h_{\boldsymbol{v}}} \langle x\rangle^{-\sigma} = \mathrm{e}^{\mathrm{i}th_{\boldsymbol{v}}} \langle x\rangle^{-\sigma} \left(\langle x\rangle^{\sigma} \mathrm{e}^{-\theta h_{\boldsymbol{v}}} \langle x\rangle^{-\sigma}\right),$$

and note that

$$\|\langle x\rangle^{\sigma} \mathrm{e}^{-\theta h_{\boldsymbol{v}}} \langle x\rangle^{-\sigma}\| = \|\mathrm{e}^{-\theta \langle x\rangle^{\sigma} h_{\boldsymbol{v}} \langle x\rangle^{-\sigma}}\| \leq \mathrm{e}^{|\theta|\, \|\langle x\rangle^{\sigma} h_{\boldsymbol{v}} \langle x\rangle^{-\sigma}\|}.$$

The desired result follows from the fact that $\langle x\rangle^{\sigma} h_{\boldsymbol{v}} \langle x\rangle^{-\sigma} = \langle x\rangle^{\sigma} h_{\mathcal{R}} \langle x\rangle^{-\sigma} + h_{\mathcal{S}} + h_{\mathrm{T}} + v_{\mathcal{R}}$ and the simple bound $\|\langle x\rangle^{\sigma} h_{\mathcal{R}} \langle x\rangle^{-\sigma}\| \leq (1 + 4^{\sigma}) c_{\mathcal{R}}$ which follows from an explicit calculation. □

Remark 1. It follows from this proof that $\|\langle x\rangle^{\sigma} \mathrm{e}^{\mathrm{i}th_{\boldsymbol{v}}} f\| \leq \|\langle x\rangle^{\sigma} \mathrm{e}^{\mathrm{i}th_{\boldsymbol{v}}} \langle x\rangle^{-\sigma}\| \, \|\langle x\rangle^{\sigma} f\| \leq \mathrm{e}^{c_{\sigma}|t|} \|\langle x\rangle^{\sigma} f\|$ so that the subspace $\mathcal{D}_{\boldsymbol{v}}$ belongs to the domain of $\langle x\rangle^{\sigma}$ for all σ.

Remark 2. In order to check Condition **(SP$_{\boldsymbol{v}}$)** it may be useful to note that

$$v_{\mathcal{S}}(E, \boldsymbol{v}) = \sum_{j=1}^m d_j^2 g_j(E, \boldsymbol{v}) \, |\phi_j\rangle\langle\phi_j|, \tag{4.4}$$

where, according to (4.3),

$$g_j(E, \boldsymbol{v}) = -\lim_{\epsilon\downarrow 0} \langle \delta_{0_j} | (h_j + v_j - E - \mathrm{i}\epsilon)^{-1} \delta_{0_j} \rangle = \begin{cases} \dfrac{1}{c_{\mathcal{R}}} \mathrm{e}^{-\mathrm{i}\theta}, & \text{for } E = v_j + 2c_{\mathcal{R}} \cos\theta, \theta \in [0, \pi]; \\ \pm \dfrac{1}{c_{\mathcal{R}}} \mathrm{e}^{-\chi}, & \text{for } E = v_j \pm 2c_{\mathcal{R}} \cosh\chi, \chi \geq 0. \end{cases} \tag{4.5}$$

Lemma 4.2. *For any $s, t \in \mathbb{R}$, $\boldsymbol{v} \in \mathbb{R}^m$ and $A \in \mathrm{CAR}(\mathfrak{h})$ one has*

$$\lim_{t_0 \to -\infty} \|\alpha_{\boldsymbol{v},t_0}^{s,t}(A) - \tau_{K_{\boldsymbol{v}}}^{t-s}(A)\| = 0.$$

Proof. We first observe that it suffices to prove the claim for the special case $A = a(f)$ for $f \in \mathfrak{h}$. Since $K_{\boldsymbol{v}}(t/t_0) - K_{\boldsymbol{v}} = -(1-\chi(t/t_0))\mathrm{d}\Gamma(b_{\boldsymbol{v}})$, with $b_{\mathbf{0}} = h_{\mathrm{T}}$ and $b_{\boldsymbol{v}} = v_{\mathcal{R}}$ for $\boldsymbol{v} \neq \mathbf{0}$, Duhamel's formula yields

$$\begin{aligned}\alpha_{\boldsymbol{v},t_0}^{s,t}(a(f)) - \tau_{K_{\boldsymbol{v}}}^{t-s}(a(f)) &= -\int_s^t \alpha_{\boldsymbol{v},t_0}^{s,u}\left(\mathrm{i}[\mathrm{d}\Gamma(b_{\boldsymbol{v}}), \tau_{K_{\boldsymbol{v}}}^{t-u}(a(f))]\right)(1-\chi(u/t_0))\mathrm{d}u \\ &= -\int_s^t \alpha_{\boldsymbol{v},t_0}^{s,u}\left(\mathrm{i}[\mathrm{d}\Gamma(b_{\boldsymbol{v}}), \Gamma_{\boldsymbol{v}}^{t-u}\tau_{H_{\boldsymbol{v}}}^{t-u}(a(f))\Gamma_{\boldsymbol{v}}^{(t-u)*}]\right)(1-\chi(u/t_0))\mathrm{d}u,\end{aligned}$$

where $\Gamma_{\boldsymbol{v}}^t = \mathrm{e}^{\mathrm{i}tK_{\boldsymbol{v}}}\mathrm{e}^{-\mathrm{i}tH_{\boldsymbol{v}}}$. One easily shows that

$$[\mathrm{d}\Gamma(b_{\boldsymbol{v}}), \Gamma_{\boldsymbol{v}}^t] = \mathrm{i}\xi\int_0^t \mathrm{e}^{\mathrm{i}sK_{\boldsymbol{v}}}[\mathrm{d}\Gamma(\mathrm{e}^{-\mathrm{i}sh_{\boldsymbol{v}}}b_{\boldsymbol{v}}\mathrm{e}^{\mathrm{i}sh_{\boldsymbol{v}}}), W]\mathrm{e}^{\mathrm{i}(t-s)K_{\boldsymbol{v}}}\mathrm{e}^{-\mathrm{i}tH_{\boldsymbol{v}}}\mathrm{d}s.$$

Since W is a polynomial in $a^\#(f)$, with $f \in \mathfrak{h}_{\mathcal{S}}$, there exists a constant C_W such that $\|[\mathrm{d}\Gamma(\mathrm{e}^{-\mathrm{i}sh_{\boldsymbol{v}}}b_{\boldsymbol{v}}\mathrm{e}^{\mathrm{i}sh_{\boldsymbol{v}}}), W]\| \leq C_W\|b_{\boldsymbol{v}}\|$ and hence we have the bound

$$\|[\mathrm{d}\Gamma(b_{\boldsymbol{v}}), \Gamma_{\boldsymbol{v}}^t]\| \leq C_W\|b_{\boldsymbol{v}}\||t\xi|.$$

This yields the estimate

$$\left\|\alpha_{\boldsymbol{v},t_0}^{s,t}(a(f)) - \tau_{K_{\boldsymbol{v}}}^{t-s}(a(f))\right\| \leq (1+2C_W|(t-s)\xi|)\,\|b_{\boldsymbol{v}}\|\,\|f\|\int_s^t (1-\chi(u/t_0))\mathrm{d}u,$$

and since $\chi : \mathbb{R} \to [0,1]$ is continuous with $\chi(0) = 1$ the result follows from the dominated convergence theorem.

□

Lemma 4.3. *Assume that Condition* (**SP**$_{\boldsymbol{v}}$) *is satisfied. If* $A \in \mathrm{CAR}_\vartheta(\mathfrak{h})$ *and* $B \in \mathrm{CAR}(\mathfrak{h})$ *are polynomials with factors in* $\{a^\#(f)\,|\,f \in \mathcal{D}_{\boldsymbol{v}}\}$*, then*

$$\int_{-\infty}^{\infty} \|[A, \tau_{H_{\boldsymbol{v}}}^t(B)]\|\mathrm{d}t < \infty.$$

Proof. The result is a direct consequence of Theorem 1.1 in [JOP3]. Nevertheless, we provide a simple and more direct proof.

It clearly suffices to consider the case where A and B are monomials. Using the CAR, one can further restrict ourselves to A's which are products of factors of the form $a^*(f_1)a(f_2)$. Finally, using the identities $[AB, C] = A[B,C] + [A,C]B$ and $[A, B^*] = -[A^*, B]^*$, it suffices to consider the case $A = a^*(f_1)a(f_2)$ and $B = a^*(g)$. Since

$$[a^*(f_1)a(f_2), \tau_{H_{\boldsymbol{v}}}^t(a^*(g))] = a^*(f_1)\langle f_2|\mathrm{e}^{\mathrm{i}th_{\boldsymbol{v}}}g\rangle,$$

the result now follows from lemma 4.1. □

4.2 Proof of Theorem 3.1

Part (1). By Remark 1 in Section 4.1 and Lemma 4.1 one has

$$C_{\boldsymbol{v}} = \sup_{t\in\mathbb{R}, f,g\in\mathcal{D}_{\boldsymbol{v}}}\left[\frac{|\langle f|\mathrm{e}^{\mathrm{i}th_{\boldsymbol{v}}}g\rangle|}{\|\langle x\rangle^3 f\|\,\|\langle x\rangle^3 g\|}\langle t\rangle^{3/2}\right] < \infty.$$

It follows from Theorem 1.2 of [JOP3] that there exists a constant C_W, depending only on the interaction W such that for $|\xi| < \bar{\xi}_{\boldsymbol{v}} = C_W/C_{\boldsymbol{v}}$, the uniform limit

$$\varsigma_{\boldsymbol{v}}(A) = \lim_{t\to\infty}\tau_{H_{\boldsymbol{v}}}^{-t}\circ\tau_{K_{\boldsymbol{v}}}^t(A), \tag{4.6}$$

exists for all $A \in \mathrm{CAR}(\mathfrak{h})$. Moreover, $\varsigma_{\boldsymbol{v}}$ is a $*$-automorphism of $\mathrm{CAR}(\mathfrak{h})$ and

$$\varsigma_{\boldsymbol{v}}^{-1}(A) = \lim_{t\to\infty} \tau_{K_{\boldsymbol{v}}}^{-t} \circ \tau_{H_{\boldsymbol{v}}}^{t}(A), \tag{4.7}$$

for all $A \in \mathrm{CAR}(\mathfrak{h})$.

Since the range of the Møller operator $\omega_{\boldsymbol{v}}$ is $\mathfrak{h}_{\mathcal{R}}$, one has

$$\mathrm{s}-\lim_{t\to\infty} \mathrm{e}^{-\mathrm{i}th_{\mathcal{R},\boldsymbol{v}}}\mathrm{e}^{\mathrm{i}th_{\boldsymbol{v}}} = \omega_{\boldsymbol{v}}.$$

It follows from the uniform continuity of the map $\mathfrak{h} \ni f \mapsto a^{\#}(f) \in \mathrm{CAR}(\mathfrak{h})$ that

$$\lim_{t\to\infty} \tau_{\mathcal{R},\boldsymbol{v}}^{-t} \circ \tau_{H_{\boldsymbol{v}}}^{t}(a^{\#}(f)) = \lim_{t\to\infty} a^{\#}(\mathrm{e}^{-\mathrm{i}th_{\mathcal{R},\boldsymbol{v}}}\mathrm{e}^{\mathrm{i}th_{\boldsymbol{v}}}f) = a^{\#}(\omega_{\boldsymbol{v}}f) = \gamma_{\omega_{\boldsymbol{v}}}(a^{\#}(f)).$$

Thus, since the maps $\tau_{\mathcal{R},\boldsymbol{v}}^{-t} \circ \tau_{H_{\boldsymbol{v}}}^{t}$ and $\gamma_{\omega_{\boldsymbol{v}}}$ are isometric $*$-morphisms,

$$\lim_{t\to\infty} \tau_{\mathcal{R},\boldsymbol{v}}^{-t} \circ \tau_{H_{\boldsymbol{v}}}^{t}(A) = \gamma_{\omega_{\boldsymbol{v}}}(A), \tag{4.8}$$

holds for any polynomial A in the $a^{\#}$, and extends by density/continuity to all $A \in \mathrm{CAR}(\mathfrak{h})$.

Combining (4.6) and (4.8) and using again the isometric nature of the various maps involved, we obtain

$$\varsigma(A) = \lim_{t\to\infty} \tau_{\mathcal{R},\boldsymbol{v}}^{-t} \circ \tau_{K_{\boldsymbol{v}}}^{t}(A) = \lim_{t\to\infty} (\tau_{\mathcal{R},\boldsymbol{v}}^{-t} \circ \tau_{H_{\boldsymbol{v}}}^{t}) \circ (\tau_{H_{\boldsymbol{v}}}^{-t} \circ \tau_{K_{\boldsymbol{v}}}^{t})(A) = \gamma_{\omega_{\boldsymbol{v}}} \circ \varsigma_{\boldsymbol{v}}(A),$$

for all $A \in \mathrm{CAR}(\mathfrak{h})$. Since ς is the composition of two $*$-isomorphisms, it is itself a $*$-isomorphism.

Part (2). Let $\langle\,\cdot\,\rangle$ be an almost-$(\boldsymbol{\beta},\boldsymbol{\mu})$-KMS state. For any $A \in \mathrm{CAR}(\mathfrak{h})$ we can write

$$\langle \tau_{K_{\boldsymbol{v}}}^{t}(A)\rangle = \langle \tau_{\mathcal{R},\boldsymbol{v}}^{t} \circ (\tau_{\mathcal{R},\boldsymbol{v}}^{-t} \circ \tau_{K_{\boldsymbol{v}}}^{t})(A)\rangle, \tag{4.9}$$

and Part (1) yields

$$\lim_{t\to\infty} \left[\langle \tau_{K_{\boldsymbol{v}}}^{t}(A)\rangle - \langle \tau_{\mathcal{R},\boldsymbol{v}}^{t}(\varsigma(A))\rangle\right] = 0.$$

Since the spectrum of $h_{\mathcal{R},\boldsymbol{v}}$ acting on $\mathfrak{h}_{\mathcal{R}}$ is purely absolutely continuous and the restriction of the state $\langle\,\cdot\,\rangle_{\mathcal{R}}^{\boldsymbol{\beta},\boldsymbol{\mu}}$ to $\mathrm{CAR}(\mathfrak{h}_{\mathcal{R}})$ is $\tau_{\mathcal{R},\boldsymbol{v}}$-invariant, the C^*-dynamical system $(\mathrm{CAR}(\mathfrak{h}_{\mathcal{R}}), \tau_{\mathcal{R},\boldsymbol{v}}, \langle\,\cdot\,\rangle_{\mathcal{R}}^{\boldsymbol{\beta},\boldsymbol{\mu}})$ is mixing (see e.g., [AJPP1, Pi]). Using the facts that $\mathrm{Ran}(\varsigma) \subset \mathrm{CAR}(\mathfrak{h}_{\mathcal{R}})$ and that the restriction to this subalgebra of the initial state $\langle\,\cdot\,\rangle$ is normal w.r.t. the restriction of $\langle\,\cdot\,\rangle_{\mathcal{R}}^{\boldsymbol{\beta},\boldsymbol{\mu}}$ we conclude that

$$\lim_{t\to\infty} \langle \tau_{K_{\boldsymbol{v}}}^{t}(A)\rangle = \lim_{t\to\infty} \langle \tau_{\mathcal{R},\boldsymbol{v}}^{t}(\varsigma(A))\rangle = \langle \varsigma(A)\rangle_{\mathcal{R}}^{\boldsymbol{\beta},\boldsymbol{\mu}}.$$

Part (3). For $A \in \mathcal{A}_{\boldsymbol{v}}$ we can invoke Theorem 1.1 in [JOP3] to conclude that the Dyson expansion (2.11) converges uniformly in t and provides a convergent expansion of the map $\varsigma_{\boldsymbol{v}}$. More precisely, one has

$$\int_{\Delta_n} \left\| \mathrm{i}[\tau_{H_{\boldsymbol{v}}}^{-s_n}(W), \mathrm{i}[\tau_{H_{\boldsymbol{v}}}^{-s_{n-1}}(W), \mathrm{i}[\cdots, \mathrm{i}[\tau_{H_{\boldsymbol{v}}}^{-s_1}(W), A]\cdots]]] \right\| \mathrm{d}s_1\cdots\mathrm{d}s_n < \infty,$$

for all $n \geq 1$ and the power series

$$\varsigma_{\boldsymbol{v}}(A) = A + \sum_{n=1}^{\infty} \xi^n \int_{\Delta_n} \mathrm{i}[\tau_{H_{\boldsymbol{v}}}^{-s_n}(W), \mathrm{i}[\tau_{H_{\boldsymbol{v}}}^{-s_{n-1}}(W), \mathrm{i}[\cdots, \mathrm{i}[\tau_{H_{\boldsymbol{v}}}^{-s_1}(W), A]\cdots]]]\mathrm{d}s_1\cdots\mathrm{d}s_n, \tag{4.10}$$

converges in norm for $|\xi| < \bar{\xi}_{\boldsymbol{v}}$. Using the fact that $\gamma_{\omega_{\boldsymbol{v}}} \circ \tau_{H_{\boldsymbol{v}}}^{t} = \tau_{\mathcal{R},\boldsymbol{v}}^{t} \circ \gamma_{\omega_{\boldsymbol{v}}}$ yields the result.

4.3 Proof of Theorem 3.2

Part (1). We claim that

$$\lim_{t_0\to-\infty} \alpha^{0,t_0}_{\mathbf{0},t_0} \circ \tau^{-t_0}_{H_\mathbf{0}}(B) = \varsigma^{-1}_{\mathbf{0}}(B), \tag{4.11}$$

holds for all $B \in \mathrm{CAR}(\mathfrak{h})$. Since $\alpha^{0,t_0}_{\mathbf{0},t_0} \circ \tau^{-t_0}_{H_\mathbf{0}}$ and $\varsigma^{-1}_{\mathbf{0}}$ are $*$-automorphisms, it suffices to prove (4.11) for $B = a(f)$ with $f \in \mathcal{D}_\mathbf{0}$. Duhamel's formula yields

$$\alpha^{0,t}_{\mathbf{0},t_0} \circ \tau^{-t}_{H_\mathbf{0}}(B) = B - \mathrm{i}\int_0^t \alpha^{0,s}_{\mathbf{0},t_0}\left([W-(1-\chi(s/t_0))H_\mathrm{T}, \tau^{-s}_{H_\mathbf{0}}(B)]\right)\mathrm{d}s,$$

$$\tau^{t}_{K_\mathbf{0}} \circ \tau^{-t}_{H_\mathbf{0}}(B) = B - \mathrm{i}\int_0^t \tau^{s}_{K_\mathbf{0}}\left([W, \tau^{-s}_{H_\mathbf{0}}(B)]\right)\mathrm{d}s.$$

Subtracting these two identities at $t = t_0 < 0$, we obtain the estimate

$$\begin{aligned}\left\|\alpha^{0,t_0}_{\mathbf{0},t_0} \circ \tau^{-t_0}_{H_\mathbf{0}}(B) - \tau^{t_0}_{K_\mathbf{0}} \circ \tau^{-t_0}_{H_\mathbf{0}}(B)\right\| &\le \int_{t_0}^0 \left\|\left(\alpha^{0,s}_{\mathbf{0},t_0} - \tau^{s}_{K_\mathbf{0}}\right)([W, \tau^{-s}_{H_\mathbf{0}}(B)])\right\|\mathrm{d}s \\ &+ \int_{t_0}^0 \left\|[H_\mathrm{T}, \tau^{-s}_{H_\mathbf{0}}(B)]\right\|(1-\chi(s/t_0))\mathrm{d}s.\end{aligned}$$

The integrand in the first term on right hand side of this expression is bounded by $2\|[W, \tau^{-s}_{H_\mathbf{0}}(B)]\|$ which is in $L^1(\mathbb{R}, \mathrm{d}s)$ by Lemma 4.3. Moreover, for fixed $s \in \mathbb{R}$, it vanishes as $t_0 \to -\infty$ by Lemma 4.2. Hence, the dominated convergence theorem allows us to conclude that the first integral vanishes as $t_0 \to -\infty$. A similar argument applies to the second integral. Taking Equ. (4.7) into account concludes the proof of our claim.

Given Equ. (4.11) we immediately obtain, for $A \in \mathrm{CAR}(\mathfrak{h})$,

$$\lim_{t_0\to-\infty}\left\|\tau^{t_0}_{H_\mathbf{0}} \circ \alpha^{t_0,0}_{\mathbf{0},t_0}(A) - \varsigma_\mathbf{0}(A)\right\| = \lim_{t_0\to-\infty}\left\|A - \alpha^{0,t_0}_{\mathbf{0},t_0} \circ \tau^{-t_0}_{H_\mathbf{0}} \circ \varsigma_\mathbf{0}(A)\right\| = \left\|A - \varsigma^{-1}_\mathbf{0} \circ \varsigma_\mathbf{0}(A)\right\| = 0.$$

From this point, the proof can proceed as for Theorem 3.1.

Part (2). Due to the fact that $V_\mathcal{R} = \mathrm{d}\Gamma(v_\mathcal{R})$ is not a local observable (and even not an element of $\mathrm{CAR}(\mathfrak{h})$) the proof is more delicate than that of Part (1).

Denote by $\alpha^{s,t}_{\mathcal{R},\boldsymbol{v},t_0}$ the non-autonomous quasi-free dynamics generated by the time-dependent one-particle Hamiltonian $h_\mathcal{R} + \chi(t/t_0)v_\mathcal{R}$. We first show that

$$\lim_{t_0\to-\infty}\left\|\alpha^{0,t_0}_{\boldsymbol{v},t_0} \circ \alpha^{t_0,0}_{\mathcal{R},\boldsymbol{v},t_0}(A) - \tau^{t_0}_{K_{\boldsymbol{v}}} \circ \tau^{-t_0}_{\mathcal{R},\boldsymbol{v}}(A)\right\| = 0, \tag{4.12}$$

holds for all $A \in \mathrm{CAR}(\mathfrak{h}_\mathcal{R})$. Since its suffices to prove this with $A = a(f)$ for a dense set of $f \in \mathfrak{h}_\mathcal{R}$, we can assume that $\langle x\rangle^\sigma f \in \mathfrak{h}_\mathcal{R}$ for some $\sigma > 5/2$. Duhamel's formula yields

$$\begin{aligned}\alpha^{0,t_0}_{\boldsymbol{v},t_0} \circ \alpha^{t_0,0}_{\mathcal{R},\boldsymbol{v},t_0}(a(f)) - \tau^{t_0}_{K_{\boldsymbol{v}}} \circ \tau^{-t_0}_{\mathcal{R},\boldsymbol{v}}(a(f)) &= \int_{t_0}^0 \tau^{s}_{K_{\boldsymbol{v}}}\left(\mathrm{i}[H_\mathrm{T}, (\tau^{-s}_{\mathcal{R},\boldsymbol{v}} - \alpha^{s,0}_{\mathcal{R},\boldsymbol{v},t_0})(a(f))]\right)\mathrm{d}s \\ &+ \int_{t_0}^0 (\tau^{s}_{K_{\boldsymbol{v}}} - \alpha^{0,s}_{\boldsymbol{v},t_0})\left(\mathrm{i}[H_\mathrm{T}, \alpha^{s,0}_{\mathcal{R},\boldsymbol{v},t_0}(a(f))]\right)\mathrm{d}s.\end{aligned} \tag{4.13}$$

Using the explicit formula

$$\alpha^{s,0}_{\mathcal{R},\boldsymbol{v},t_0}(a(f)) = a\left(\mathrm{e}^{\mathrm{i}\int_s^0 \chi(u/t_0)v_\mathcal{R}\mathrm{d}u}\mathrm{e}^{-\mathrm{i}sh_\mathcal{R}}f\right), \tag{4.14}$$

the CAR and Lemma 4.3, one derives the estimate

$$\left\|[H_{\rm T},(\tau_{\mathcal{R},\boldsymbol{v}}^{-s}-\alpha_{\mathcal{R},\boldsymbol{v},t_0}^{s,0})(a(f))]\right\|\le C_\sigma\sum_{j=1}^m|d_j|\,\|\langle x\rangle^\sigma f\|\,|1-{\rm e}^{{\rm i}v_j\int_s^0(1-\chi(u/t_0)){\rm d}u}|\,\langle s\rangle^{-3/2}.$$

It follows from the dominated convergence theorem that the first integral on the right hand side of Equ. (4.13) vanishes as $t_0\to-\infty$. Due to Lemma 4.2 and the estimate

$$\left\|[H_{\rm T},\alpha_{\mathcal{R},\boldsymbol{v},t_0}^{s,0}(a(f))]\right\|\le C_\sigma\sum_{j=1}^m|d_j|\,\|\langle x\rangle^\sigma f\|\,\langle s\rangle^{-3/2},$$

the same is true for the second integral and this proves (4.12).

Since the range of $\gamma_{\omega_{\boldsymbol{v}}}$ is ${\rm CAR}(\mathfrak{h}_\mathcal{R})$, we have

$$\begin{aligned}\lim_{t_0\to-\infty}\left\|\alpha_{\mathcal{R},\boldsymbol{v},t_0}^{0,t_0}\circ\alpha_{\boldsymbol{v},t_0}^{t_0,0}(A)-\gamma_{\omega_{\boldsymbol{v}}}\circ\varsigma_{\boldsymbol{v}}(A)\right\|&=\lim_{t_0\to-\infty}\left\|A-\alpha_{\boldsymbol{v},t_0}^{0,t_0}\circ\alpha_{\mathcal{R},\boldsymbol{v},t_0}^{t_0,0}(\gamma_{\omega_{\boldsymbol{v}}}\circ\varsigma_{\boldsymbol{v}}(A))\right\|\\&=\lim_{t_0\to-\infty}\left\|A-\tau_{K_{\boldsymbol{v}}}^{t_0}\circ\tau_{\mathcal{R},\boldsymbol{v}}^{-t_0}(\gamma_{\omega_{\boldsymbol{v}}}\circ\varsigma_{\boldsymbol{v}}(A))\right\|\\&=\lim_{t_0\to-\infty}\left\|\tau_{\mathcal{R},\boldsymbol{v}}^{t_0}\circ\tau_{K_{\boldsymbol{v}}}^{-t_0}(A)-\gamma_{\omega_{\boldsymbol{v}}}\circ\varsigma_{\boldsymbol{v}}(A)\right\|\\&=0,\end{aligned}$$

for any $A\in{\rm CAR}(\mathfrak{h})$. Now it follows from (4.14) that $\alpha_{\mathcal{R},\boldsymbol{v},t_0}^{0,t_0}=\tau_{\mathcal{R},\tilde{\boldsymbol{v}}}^{t_0}$ with

$$\tilde{\boldsymbol{v}}=\boldsymbol{v}\int_0^1\chi(s){\rm d}s,$$

so that we can write the following analogue of Equ. (4.9)

$$\langle\alpha_{\boldsymbol{v},t_0}^{t_0,0}(A)\rangle=\langle\tau_{\mathcal{R},\tilde{\boldsymbol{v}}}^{-t_0}\circ(\alpha_{\mathcal{R},\boldsymbol{v},t_0}^{0,t_0}\circ\alpha_{\boldsymbol{v},t_0}^{t_0,0})(A)\rangle,$$

and finish the proof as for Theorem 3.1.

4.4 Proof of Theorem 3.3

The main arguments used in this section are simple adaptations of Section 3.6 and 3.7 in [JOP2].

Part (1). We start with some basic facts from modular theory (the reader is referred to [BR1] for a detailed exposition). A state on ${\rm CAR}(\mathfrak{h})$ is called modular if it is a (β,μ)-KMS state with $\beta=-1$ and $\mu=0$ for some strongly continuous group σ of $*$-automorphisms of ${\rm CAR}(\mathfrak{h})$[1]. The state $\langle\,\cdot\,\rangle_\mathcal{R}^{\boldsymbol{\beta},\boldsymbol{\mu}}$ is modular and its modular group σ is easily seen to be the quasi-free dynamics generated by the one-particle Hamiltonian

$$k=-\sum_{j=1}^m\beta_j(h_j-\mu_j1_j).$$

We denote by δ the generator of σ, i.e., $\sigma^t={\rm e}^{t\delta}$.

For a self-adjoint $P\in{\rm CAR}_\vartheta(\mathfrak{h})$ define the group σ_P by $\sigma_P^t(A)={\rm e}^{{\rm i}t({\rm d}\Gamma(k)+P)}A{\rm e}^{-{\rm i}t({\rm d}\Gamma(k)+P)}$. By Araki's perturbation theory σ_P has a unique $(-1,0)$-KMS state which we denote $\langle\,\cdot\,\rangle_P$. Let $\mathfrak{S}$ be the set of all states

[1] The choice of $\beta=-1$ and $\mu=0$ is conventional in the mathematical literature and has no physical meaning

obtained in this way. This set is norm dense in the set of $\langle\,\cdot\,\rangle_{\mathcal{R}}^{\boldsymbol{\beta},\boldsymbol{\mu}}$-normal states on $\mathrm{CAR}_\vartheta(\mathfrak{h})$ (this is a consequence of the final remark in Section 5 of [A1]).

The fact that $\mathrm{S}(\langle\,\cdot\,\rangle_P|\langle\,\cdot\,\rangle_P) = 0$ and the fundamental formula of Araki (Theorem 3.10 in [A2] or Proposition 6.2.32 in [BR2]) yield

$$\mathrm{S}(\langle\tau_{K_{\boldsymbol{v}}}^t(\,\cdot\,)\rangle_P|\langle\,\cdot\,\rangle_P) = \mathrm{S}(\langle\tau_{K_{\boldsymbol{v}}}^t(\,\cdot\,)\rangle_P|\langle\,\cdot\,\rangle_{\mathcal{R}}^{\beta,\mu}) - \mathrm{S}(\langle\,\cdot\,\rangle_P|\langle\,\cdot\,\rangle_{\mathcal{R}}^{\beta,\mu}) + \langle\tau_{K_{\boldsymbol{v}}}^t(P) - P\rangle_P. \tag{4.15}$$

Let $Q = H_{\mathcal{S}} + \xi W + H_{\mathrm{T}} \in \mathrm{CAR}_\vartheta(\mathfrak{h}_{\mathcal{S}})$ and note that $K_{\boldsymbol{v}} = H_{\mathcal{R},\boldsymbol{v}} + Q$. A simple calculation shows that $Q \in \mathrm{Dom}(\delta)$ and

$$\delta(Q) = \mathrm{i}[\mathrm{d}\Gamma(k), Q] = \mathrm{i}[-\sum_{j=1}^m \beta_j(H_j - \mu_j N_j), K_v - H_{\mathcal{R},\boldsymbol{v}}] = -\sum_{j=1}^m \beta_j(E_j - \mu_j J_j). \tag{4.16}$$

Since $\langle\,\cdot\,\rangle_{\mathcal{R}}^{\beta,\mu}$ is $\tau_{\mathcal{R},\boldsymbol{v}}$-invariant, we can apply the entropy balance formula of [JP3, JP4] to obtain

$$\mathrm{S}(\langle\tau_{K_{\boldsymbol{v}}}^t(\,\cdot\,)\rangle_P|\langle\,\cdot\,\rangle_{\mathcal{R}}^{\beta,\mu}) = \mathrm{S}(\langle\,\cdot\,\rangle_P|\langle\,\cdot\,\rangle_{\mathcal{R}}^{\beta,\mu}) - \int_0^t \langle\tau_{K_{\boldsymbol{v}}}^s(\delta(Q))\rangle_P\,\mathrm{d}s.$$

Inserting this relation into (4.15) further gives

$$\mathrm{S}(\langle\tau_{K_v}^{-t_0}(\,\cdot\,)\rangle_P|\langle\,\cdot\,\rangle_P) = \langle\tau_{K_v}^{-t_0}(P) - P\rangle_P - \int_0^{-t_0} \langle\tau_{K_v}^s(\delta(Q))\rangle_P\,\mathrm{d}s. \tag{4.17}$$

Dividing by t_0 and taking the limit $t_0 \to -\infty$ we get

$$\lim_{t_0\to-\infty} \frac{1}{t_0}\mathrm{S}(\langle\tau_{K_v}^{-t_0}(\,\cdot\,)\rangle_P|\langle\,\cdot\,\rangle_P) = \langle\delta(Q)\rangle_+^{\boldsymbol{\beta},\boldsymbol{\mu},\boldsymbol{v}},$$

which, taking into account (4.16) and the fact that relative entropies are non-positive, yields the result.

Part (2). This is just a special case of Part (1). It suffices to notice that if $\boldsymbol{\beta} = (\beta,\ldots,\beta)$ and $\boldsymbol{\mu} = (\mu,\ldots,\mu)$ then $k = -\beta(h_{\mathcal{R}} - \mu)$ and hence $\mathrm{d}\Gamma(k) + P = -\beta(K_{\mathbf{0}} - \mu N)$ where $P = -\beta(Q - \mu N_{\mathcal{S}}) \in \mathrm{CAR}_\vartheta(\mathfrak{h})$. It follows that $\langle\,\cdot\,\rangle_{K_{\mathbf{0}}}^{\beta,\mu} = \langle\,\cdot\,\rangle_P \in \mathfrak{S}$. Finally, taking into account the sum rules (2.8) we get

$$-\sum_{j=1}^m \beta\langle E_j - \mu J_j\rangle_+^{\boldsymbol{\beta},\boldsymbol{\mu},\boldsymbol{v}} = \sum_{j=1}^m \beta v_j\langle J_j\rangle_+^{\boldsymbol{\beta},\boldsymbol{\mu},\boldsymbol{v}}.$$

We note for later reference that Equ. (4.17) yields the following entropy balance relation for the partition-free NESS

$$\mathrm{S}(\langle\tau_{K_v}^t(\,\cdot\,)\rangle_{K_{\mathbf{0}}}^{\beta,\mu}|\langle\,\cdot\,\rangle_{K_{\mathbf{0}}}^{\beta,\mu}) = \langle\tau_{K_v}^t(P) - P\rangle_{K_{\mathbf{0}}}^{\beta,\mu} - \int_0^t \langle\tau_{K_v}^s(\delta(Q))\rangle_{K_{\mathbf{0}}}^{\beta,\mu}\,\mathrm{d}s. \tag{4.18}$$

Part (3). Let $\boldsymbol{\beta} = (\beta,\ldots,\beta)$, $\boldsymbol{\mu} = (\mu,\ldots,\mu)$ and $\boldsymbol{v}_0 = (v_0,\ldots,v_0)$ and set $\widetilde{K}_{\mathbf{0}} = K_{\mathbf{0}} - v_0 N_{\mathcal{S}}$. It follows from the identity

$$K_{\boldsymbol{v}_0} = K_{\mathbf{0}} + v_0\sum_{j=1}^m N_j = \widetilde{K}_{\mathbf{0}} + v_0 N,$$

that $\tau_{K_{\boldsymbol{v}_0}}^t = \tau_{\widetilde{K}_{\mathbf{0}}}^t \circ \vartheta^{tv_0}$. The gauge invariance $\vartheta^s \circ \tau_{\mathcal{R},\boldsymbol{v}_0}^{-t} \circ \tau_{K_{\boldsymbol{v}_0}}^t = \tau_{\mathcal{R},\boldsymbol{v}_0}^{-t} \circ \tau_{K_{\boldsymbol{v}_0}}^t \circ \vartheta^s$ and the fact that ς maps onto $\mathrm{CAR}(\mathfrak{h}_{\mathcal{R}})$ implies $\varsigma \circ \vartheta^t = \vartheta_{\mathcal{R}}^t \circ \varsigma$ where $\vartheta_{\mathcal{R}}$ is the gauge group of $\mathrm{CAR}(\mathfrak{h}_{\mathcal{R}})$ (i.e., the quasi-free dynamics on $\mathrm{CAR}(\mathfrak{h}_{\mathcal{R}})$ generated by $1_{\mathcal{R}}$). Together with the intertwining property of ς, this yields

$$\varsigma \circ (\tau_{\widetilde{K}_{\mathbf{0}}}^t \circ \vartheta^{-t\mu}) = \varsigma \circ \tau_{K_{\boldsymbol{v}_0}} \circ \vartheta^{-t(\mu+v_0)} = \tau_{\mathcal{R},\boldsymbol{v}_0}^t \circ \varsigma \circ \vartheta^{-t(\mu+v_0)} = (\tau_{\mathcal{R}}^t \circ \vartheta_{\mathcal{R}}^{-t\mu}) \circ \varsigma.$$

It follows that for any $A, B \in \mathrm{CAR}(\mathfrak{h})$ one has

$$\langle A\,\tau^t_{\widetilde{K}_{\mathbf{0}}} \circ \vartheta^{-t\mu}(B)\rangle_+^{\boldsymbol{\beta},\boldsymbol{\mu},\boldsymbol{v}_0} = \langle \varsigma(A)\varsigma(\tau^t_{\widetilde{K}_{\mathbf{0}}} \circ \vartheta^{-t\mu}(B))\rangle_{\mathcal{R}}^{\boldsymbol{\beta},\boldsymbol{\mu}} = \langle \varsigma(A)\tau^t_{\mathcal{R}} \circ \vartheta^{-t\mu}_{\mathcal{R}}(\varsigma(B))\rangle_{\mathcal{R}}^{\boldsymbol{\beta},\boldsymbol{\mu}},$$

and since $\langle\,\cdot\,\rangle_{\mathcal{R}}^{\boldsymbol{\beta},\boldsymbol{\mu}}$ is (β,μ)-KMS for $\tau_{\mathcal{R}}$ one easily concludes that $\langle\,\cdot\,\rangle_+^{\boldsymbol{\beta},\boldsymbol{\mu},\boldsymbol{v}_0}$ satisfies the (β,μ)-KMS condition for $\tau_{\widetilde{K}_{\mathbf{0}}}$. Since the later group is a local perturbation of the quasi-free dynamics $\tau_{H_{\mathbf{0}}}$, it follows from Araki's perturbation theory that the partition-free NESS is the unique (β,μ)-KMS state for $\tau_{\widetilde{K}_{\mathbf{0}}}$ which we denote $\langle\,\cdot\,\rangle_{\widetilde{K}_{\mathbf{0}}}^{\beta,\mu}$.

Observe that replacing the one-particle Hamiltonian $h_{\mathcal{S}}$ by $h_{\mathcal{S}} - v_0 1_{\mathcal{S}}$ transforms $K_{\mathbf{0}}$ into $\widetilde{K}_{\mathbf{0}}$. The same substitution changes $K_{\boldsymbol{v}}$ into $\widetilde{K}_{\boldsymbol{v}} = K_{\boldsymbol{v}_0+\boldsymbol{v}} - v_0 N = K_v - v_0 N_{\mathcal{S}}$ while the entropy balance relation (4.18) transforms into

$$\mathrm{S}(\langle \tau^t_{\widetilde{K}_{\boldsymbol{v}}}(\,\cdot\,)\rangle_{\widetilde{K}_{\mathbf{0}}}^{\beta,\mu} | \langle\,\cdot\,\rangle_{\widetilde{K}_{\mathbf{0}}}^{\beta,\mu}) = \langle \tau^t_{\widetilde{K}_{\boldsymbol{v}}}(\widetilde{P}) - \widetilde{P}\rangle_{\widetilde{K}_{\mathbf{0}}}^{\beta,\mu} - \int_0^t \langle \tau^s_{\widetilde{K}_{\boldsymbol{v}}}(\delta(\widetilde{Q}))\rangle_{\widetilde{K}_{\mathbf{0}}}^{\beta,\mu}\,\mathrm{d}s,$$

where $\widetilde{Q} = Q - v_0 N_{\mathcal{S}}$ and $\widetilde{P} = P + \beta v_0 N_{\mathcal{S}}$. Dividing this relation by $t > 0$ and letting $t \downarrow 0$ we obtain, after some elementary algebra and using the fact that relative entropies are non-positive

$$0 \geq \lim_{t\downarrow 0} \frac{1}{t}\mathrm{S}(\langle \tau^t_{\widetilde{K}_{\boldsymbol{v}}}(\,\cdot\,)\rangle_{\widetilde{K}_{\mathbf{0}}}^{\beta,\mu} | \langle\,\cdot\,\rangle_{\widetilde{K}_{\mathbf{0}}}^{\beta,\mu}) = \langle \mathrm{i}[\widetilde{K}_{\boldsymbol{v}}, \widetilde{P}] - \delta(\widetilde{Q})\rangle_{\widetilde{K}_{\mathbf{0}}}^{\beta,\mu} = -\beta\sum_{j=1}^m v_j \langle J_j\rangle_{\widetilde{K}_{\mathbf{0}}}^{\beta,\mu}.$$

Since this relation holds for all $\boldsymbol{v} \in \mathbb{R}^m$ and $\langle J_j\rangle_{\widetilde{K}_{\mathbf{0}}}^{\beta,\mu}$ does not depend on $\boldsymbol{v}$ we can conclude that $\langle J_j\rangle_{\widetilde{K}_{\mathbf{0}}}^{\beta,\mu} = 0$ for all j.

To deal with the energy currents, we set $K^\natural_{\boldsymbol{\alpha}} = \widetilde{K}_{\mathbf{0}} + \sum_{j=1}^m \alpha_j H_j$ with $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^m$ and invoke the same arguments to derive the inequality

$$0 \geq \lim_{t\downarrow 0} \frac{1}{t}\mathrm{S}(\langle \tau^t_{K^\natural_{\boldsymbol{\alpha}}}(\,\cdot\,)\rangle_{\widetilde{K}_{\mathbf{0}}}^{\beta,\mu} | \langle\,\cdot\,\rangle_{\widetilde{K}_{\mathbf{0}}}^{\beta,\mu}) = \langle \mathrm{i}[K^\natural_{\boldsymbol{\alpha}}, \widetilde{P}] - \delta(\widetilde{Q})\rangle_{\widetilde{K}_{\mathbf{0}}}^{\beta,\mu} = -\beta\sum_{j=1}^m \alpha_j \langle E_j\rangle_{\widetilde{K}_{\mathbf{0}}}^{\beta,\mu},$$

from which we conclude that $\langle E_j\rangle_{\widetilde{K}_{\mathbf{0}}}^{\beta,\mu} = 0$ for all j.

4.5 Proof of Theorem 3.4

We only consider the lesser Green-Keldysh function. The case of the greater function is completely similar.

Parts (1)–(3). We observe that $A_x = \varsigma(a_x)$ where the Møller morphism ς is given by $\varsigma = \gamma_{\omega_{\boldsymbol{v}}} \circ \varsigma_{\boldsymbol{v}}$. Since the range of $\gamma_{\omega_{\boldsymbol{v}}}$ is $\mathrm{CAR}(\mathfrak{h}_{\mathcal{R}})$ one has $A_x \in \mathrm{CAR}(\mathfrak{h}_{\mathcal{R}})$. Using the intertwining property of ς, Theorem 3.1 allows us to write

$$G_+^{<\boldsymbol{\beta},\boldsymbol{\mu},\boldsymbol{v}}(t;x,y) = \mathrm{i}\langle a_y^*\tau^t_{K_{\boldsymbol{v}}}(a_x)\rangle_+^{\boldsymbol{\beta},\boldsymbol{\mu},\boldsymbol{v}} = \mathrm{i}\langle \varsigma(a_y^*\tau^t_{K_{\boldsymbol{v}}}(a_x))\rangle_{\mathcal{R}}^{\boldsymbol{\beta},\boldsymbol{\mu}} = \mathrm{i}\langle \varsigma(a_y^*)\tau^t_{\mathcal{R},\boldsymbol{v}}(\varsigma(a_x))\rangle_{\mathcal{R}}^{\boldsymbol{\beta},\boldsymbol{\mu}} = \mathrm{i}\langle A_y^*\tau^t_{\mathcal{R},\boldsymbol{v}}(A_x)\rangle_{\mathcal{R}}^{\boldsymbol{\beta},\boldsymbol{\mu}}.$$

Passing to the GNS representation and using the fact that $L_{\mathcal{R},\boldsymbol{v}}\Omega_{\mathcal{R}} = 0$ we obtain

$$G_+^{<\boldsymbol{\beta},\boldsymbol{\mu},\boldsymbol{v}}(t;x,y) = \mathrm{i}(\Omega_{\mathcal{R}}|\pi_{\mathcal{R}}(A_y)^*\mathrm{e}^{\mathrm{i}tL_{\mathcal{R},\boldsymbol{v}}}\pi_{\mathcal{R}}(A_x)\mathrm{e}^{-\mathrm{i}tL_{\mathcal{R},\boldsymbol{v}}}\Omega_{\mathcal{R}}) = \mathrm{i}(\Psi_y|\mathrm{e}^{\mathrm{i}tL_{\mathcal{R},\boldsymbol{v}}}\Psi_x) = \mathrm{i}\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}t\omega}\mathrm{d}\lambda_{\Psi_y,\Psi_x}(\omega),$$

where $\Psi_x = \pi_{\mathcal{R}}(A_x)\Omega_{\mathcal{R}}$ and λ_{Ψ_y,Ψ_x} denotes the spectral measure of $L_{\mathcal{R},\boldsymbol{v}}$ for Ψ_y and Ψ_x. We note that

$$(\Omega_{\mathcal{R}}|\Psi_x) = (\Omega_{\mathcal{R}}|\pi_{\mathcal{R}}(A_x)\Omega_{\mathcal{R}}) = \langle A_x\rangle_{\mathcal{R}}^{\boldsymbol{\beta},\boldsymbol{\mu}} = \langle a_x\rangle_+^{\boldsymbol{\beta},\boldsymbol{\mu},\boldsymbol{v}} = 0,$$

since the NESS $\langle\,\cdot\,\rangle_+^{\boldsymbol{\beta},\boldsymbol{\mu},\boldsymbol{v}}$ is gauge-invariant. This proves Part (1). Moreover, this implies that the spectral measure λ_{Ψ_x,Ψ_y} is absolutely continuous w.r.t. Lebesgue's measure so that

$$G_+^{<\boldsymbol{\beta},\boldsymbol{\mu},\boldsymbol{v}}(t;x,y) = \mathrm{i}\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}t\omega}\frac{\mathrm{d}\lambda_{\Psi_y,\Psi_x}(\omega)}{\mathrm{d}\omega}\,\mathrm{d}\omega,$$

which proves Part (2).

Part (4). We first note that a_x is an entire analytic element for the group $\tau_{K_{\boldsymbol{v}}}$. This is an simple consequence of the interaction picture representation

$$\tau_{K_{\boldsymbol{v}}}^t(a_x) = \Gamma_{\boldsymbol{v}}^t \tau_{H_{\boldsymbol{v}}}^t(a_x)\Gamma_{\boldsymbol{v}}^{t*},$$

where the cocycle $\Gamma_{\boldsymbol{v}}^t = \mathrm{e}^{\mathrm{i}tK_{\boldsymbol{v}}}\mathrm{e}^{-\mathrm{i}tH_{\boldsymbol{v}}}$ has the Dyson expansion

$$\Gamma_{\boldsymbol{v}}^t = I + \sum_{n=1}^{\infty}(\mathrm{i}\xi t)^n \int\limits_{0\le s_1\le\cdots\le s_n\le 1} \tau_{H_{\boldsymbol{v}}}^{ts_1}(W)\cdots\tau_{H_{\boldsymbol{v}}}^{ts_n}(W)\mathrm{d}s_1\cdots\mathrm{d}s_n.$$

Indeed, since $h_{\boldsymbol{v}}$ is bounded, $\tau_{H_{\boldsymbol{v}}}^t(a_x) = a(\mathrm{e}^{\mathrm{i}th_{\boldsymbol{v}}}\delta_x)$ and $\tau_{H_{\boldsymbol{v}}}^t(W)$ extend to entire analytic functions of t and the above Dyson expansion converge in norm for any complex value of t, defining an entire analytic $\mathrm{CAR}(\mathfrak{h})$-valued function. Finally, since

$$\partial_z^n G_+^{<\boldsymbol{\beta},\boldsymbol{\mu},\boldsymbol{v}}(z+t;x,y) = \mathrm{i}\langle a_y^*\tau_{K_{\boldsymbol{v}}}^t(A_z)\rangle_+^{\boldsymbol{\beta},\boldsymbol{\mu},\boldsymbol{v}},$$

where $A_z = \partial_z^n\tau_{K_{\boldsymbol{v}}}^z(a_x)\in\mathrm{CAR}(\mathfrak{h})$, the last assertion follows from the mixing property of the dynamical system $(\mathrm{CAR}(\mathfrak{h}),\tau_{K_{\boldsymbol{v}}},\langle\,\cdot\,\rangle_+^{\boldsymbol{\beta},\boldsymbol{\mu},\boldsymbol{v}})$ (Remark 4, Section 3.1).

Parts (5)–(6). By Theorem 1.1 of [JOP3] the Dyson expansion (4.10) can be reorganized as

$$\varsigma_{\boldsymbol{v}}(a_x) = \sum_{n=0}^{\infty}\xi^n\sum_{q\in\mathcal{Q}_n}\int_{\Delta_n} G_{x,q}^{(n)}(s)F_q^{(n)}(s)\mathrm{d}s,$$

where the $\mathcal{Q}_n$ are finite sets, the $G_{x,q}^{(n)}(s)$ scalar functions of $s=(s_1,\ldots,s_n)\in\mathbb{R}^n$ such that

$$\sum_{n=0}^{\infty}\bar{\xi}_{\boldsymbol{v}}^n\sum_{q\in\mathcal{Q}_n}\int_{\Delta_n}|G_{x,q}^{(n)}(s)|\mathrm{d}s<\infty, \tag{4.19}$$

and the $F_q^{(n)}(s)$ are monomials of degree $k_q^{(n)}$ with factors in $\{a^\#(\mathrm{e}^{-\mathrm{i}uh_{\boldsymbol{v}}}\delta_z)\,|\,z\in\mathcal{S}, u\in\{s_1,\ldots,s_n\}\}$. Moreover, the $k_q^{(n)}$ are odd and satisfy $k_q^{(n)}\le nk_W+1$ for some integer k_W depending only on the interaction W.

Setting $\langle\,\cdot\,\rangle_+ = \langle\,\cdot\,\rangle_{+,\xi=0}^{\boldsymbol{\beta},\boldsymbol{\mu},\boldsymbol{v}}$, the identity $\langle a_y^*\tau_{K_{\boldsymbol{v}}}^t(a_x)\rangle_+^{\boldsymbol{\beta},\boldsymbol{\mu},\boldsymbol{v}} = \langle\varsigma_{\boldsymbol{v}}(a_y^*)\tau_{H_{\boldsymbol{v}}}^t(\varsigma_{\boldsymbol{v}}(a_x))\rangle_+$ leads to the estimate

$$\begin{aligned}\int_{-\infty}^{\infty}\left|G_+^{<\boldsymbol{\beta},\boldsymbol{\mu},\boldsymbol{v}}(t+\mathrm{i}\eta,x,y)\right|\,dt \le \sum_{n_1,n_2=0}^{\infty}|\xi|^{n_1+n_2}\sum_{\substack{q_1\in\mathcal{Q}_{n_1}\\ q_2\in\mathcal{Q}_{n_2}}}&\left[\int\limits_{\Delta_{n_1}}|G_{y,q_1}^{(n_1)}(s)|\mathrm{d}s\right]\left[\int\limits_{\Delta_{n_2}}|G_{x,q_2}^{(n_2)}(s')|\mathrm{d}s'\right]\\ &\times\sup_{(s,s')\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}}\int_{-\infty}^{\infty}|\langle F_{q_1}^{(n_1)}(s)^*\tau_{H_{\boldsymbol{v}}}^{t+\mathrm{i}\eta}(F_{q_2}^{(n_2)}(s'))\rangle_+|\mathrm{d}t.\end{aligned}\tag{4.20}$$

To estimate the last integral we denote by k_j the degree of the monomial $F_{q_j}^{(n_j)}$ and expand each factor of this monomial in terms of field operators using the identity $a^\#(f) = 2^{-1/2}(\varphi(f) \pm \mathrm{i}\varphi(\mathrm{i}f))$. In this way we can write $\langle F_{q_1}^{(n_1)}(s)^* \tau_{H_{\boldsymbol{v}}}^{t+\mathrm{i}\eta}(F_{q_2}^{(n_2)}(s'))\rangle_+$ as a sum of $2^{k_1+k_2}$ terms of the form $2^{-(k_1+k_2)/2}\langle\varphi(f_1)\cdots\varphi(f_{k_1+k_2})\rangle_+$, where

$$f_i \in \begin{cases} \{\mathrm{e}^{-\mathrm{i}uh_{\boldsymbol{v}}}\delta_z \,|\, z \in \mathcal{S}, u \in \{s_1, \ldots, s_{n_1}\}\} & \text{if} \quad i \in \mathcal{V}_1 = \{1, \ldots, k_1\}, \\ \{\mathrm{e}^{\mathrm{i}(t-u'\pm\mathrm{i}\eta)h_{\boldsymbol{v}}}\delta_z \,|\, z \in \mathcal{S}, u' \in \{s'_1, \ldots, s'_{n_2}\}\} & \text{if} \quad i \in \mathcal{V}_2 = \{k_1+1, \ldots, k_1+k_2\}. \end{cases}$$

Since k_1 and k_2 are odd, each term in the Wick expansion (2.4) of $\langle\varphi(f_1)\cdots\varphi(f_{k_1+k_2})\rangle_+$ contains a factor $\langle\varphi(f_i)\varphi(f_j)\rangle_+$ such that $i \in \mathcal{V}_1$ and $j \in \mathcal{V}_2$. For such a pair (i,j) denote by $\mathcal{P}_{i,j}$ the set of pairings of elements of $(\mathcal{V}_1 \cup \mathcal{V}_2) \setminus \{i,j\}$ and for $p \in \mathcal{P}_{i,j}$ let $p \vee (i,j)$ be the pairing on $\mathcal{V}_1 \cup \mathcal{V}_2$ obtained by merging p with the pair (i,j). Invoking Lemma 4.1 of [JPP] we can write

$$\begin{aligned}
\langle\varphi(f_1)\cdots\varphi(f_{k_1+k_2})\rangle_+ &= \sum_{i\in\mathcal{V}_1}\sum_{j\in\mathcal{V}_2}\langle\varphi(f_i)\varphi(f_j)\rangle_+ \sum_{p\in\mathcal{P}_{i,j}} \varepsilon(p\vee(i,j)) \prod_{(k,l)\in p}\langle\varphi(f_k)\varphi(f_l)\rangle_+ \\
&= \sum_{i\in\mathcal{V}_1}\sum_{j\in\mathcal{V}_2}\varepsilon_{i,j}\langle\varphi(f_i)\varphi(f_j)\rangle_+ \sum_{p\in\mathcal{P}_{i,j}} \varepsilon(p) \prod_{(k,l)\in p}\langle\varphi(f_k)\varphi(f_l)\rangle_+ \\
&= \sum_{i\in\mathcal{V}_1}\sum_{j\in\mathcal{V}_2}\varepsilon_{i,j}\langle\varphi(f_i)\varphi(f_j)\rangle_+\langle\varphi(f_1)\cdots\cancel{\varphi(f_i)}\cdots\cancel{\varphi(f_j)}\cdots\varphi(f_{k_1+k_2})\rangle_+,
\end{aligned}$$

where $|\varepsilon_{i,j}| = 1$. Using the facts that $\|\varphi(f)\| = 2^{-1/2}\|f\|$, $\langle\varphi(f_i)\varphi(f_j)\rangle_\varrho = 2^{-1}(\langle f_i|(I-\varrho)f_j\rangle + \langle f_j|\varrho f_i\rangle)$ and Lemma 4.1 we get the estimate

$$|\langle\varphi(f_i)\varphi(f_j)\rangle_+\langle\varphi(f_1)\cdots\cancel{\varphi(f_i)}\ldots\cancel{\varphi(f_j)}\cdots\varphi(f_{k_1+k_2})\rangle_+| \le C\mathrm{e}^{c|\eta|k_2}2^{-(k_1+k_2)/2}\langle t-u\rangle^{-3/2}$$

for some constants C and c and some $u \in \mathbb{R}$. Integrating over t before summing all contributions yields, taking into account the bounds $k_j \le n_j k_W + 1$,

$$\int_{-\infty}^{\infty} |\langle F_{q_1}^{(n_1)}(s)^* \tau_{H_{\boldsymbol{v}}}^{t+\mathrm{i}\eta}(F_{q_2}^{(n_2)}(s'))\rangle_+|\mathrm{d}t \le C'\mathrm{e}^{c'|\eta|n_2}n_1n_2,$$

for some constants C' and c'. The estimate (4.19) allows us to conclude that the integral on the left-hand side of (4.20) is finite provided $|\xi|\mathrm{e}^{c'|\eta|} < \bar{\xi}_{\boldsymbol{v}}$ so that Part (5) holds for any $\theta < \log(\bar{\xi}_{\boldsymbol{v}}/|\xi|)/c'$. Part (6) follows from a Paley-Wiener argument (see, e.g., Theorem IX.14 in [RS2]).

4.6 Proof of Theorem 3.5

Part (1). Note that the function $\mathbb{R} \ni E \mapsto v_{\mathcal{S}}(E, \boldsymbol{v})$ is continuous and vanishes at infinity. Thus, if Assumption **(SP$_{\boldsymbol{v}}$)** is satisfied then $C_{\mathcal{S}} = \sup_{E\in\mathbb{R}}\|\mathfrak{m}_{\boldsymbol{v}}(E)\| < \infty$. Since the matrix

$$\mathfrak{m}_{\mathrm{HF},\boldsymbol{v}}(E) = (h_{\mathcal{S}} + \xi v_{\mathrm{HF}} + v_{\mathcal{S}}(E,\boldsymbol{v}) - E)^{-1},$$

exists for all $E \in \mathbb{R}$ if $|\xi| < (C_{\mathcal{S}}\|v_{\mathrm{HF}}\|)^{-1}$, Part (1) follows from Theorem 3.1.

Part (2). To simplify notation let $\langle\,\cdot\,\rangle_+$ denote the non-interacting NESS $\langle\,\cdot\,\rangle_{+,\xi=0}^{\boldsymbol{\beta},\boldsymbol{\mu},\boldsymbol{v}}$. Starting with the fact that $\langle A\tau_{K_{\boldsymbol{v}}}^t(B)\rangle_+^{\boldsymbol{\beta},\boldsymbol{\mu},\boldsymbol{v}} = \langle\varsigma_{\boldsymbol{v}}(A)\tau_{H_{\boldsymbol{v}}}^t(\varsigma_{\boldsymbol{v}}(B))\rangle_+$ and using the $\tau_{H_{\boldsymbol{v}}}$-invariance of $\langle\,\cdot\,\rangle_+$, the Dyson expansion (4.10) yields

$$\langle A\tau_{K_{\boldsymbol{v}}}^t(B)\rangle_+^{\boldsymbol{\beta},\boldsymbol{\mu},\boldsymbol{v}} = \langle A\tau_{H_{\boldsymbol{v}}}^t(B)\rangle_+ + \mathrm{i}\xi\int_0^\infty \langle[W,\tau_{H_{\boldsymbol{v}}}^s(A)]\tau_{H_{\boldsymbol{v}}}^{s+t}(B) + \tau_{H_{\boldsymbol{v}}}^{s-t}(A)[W,\tau_{H_{\boldsymbol{v}}}^s(B)]\rangle_+\,\mathrm{d}s + \mathcal{O}(\xi^2),$$

for $A, B \in \mathcal{A}_{\boldsymbol{v}}$. In the same way, we get

$$\langle A\tau^t_{K_{\boldsymbol{v}}}(B)\rangle^{\boldsymbol{\beta},\boldsymbol{\mu},\boldsymbol{v}}_{\mathrm{HF}+} = \langle A\tau^t_{H_{\boldsymbol{v}}}(B)\rangle_+ + \mathrm{i}\xi\int_0^\infty \langle [W_{\mathrm{HF}}, \tau^s_{H_{\boldsymbol{v}}}(A)]\tau^{s+t}_{H_{\boldsymbol{v}}}(B) + \tau^{s-t}_{H_{\boldsymbol{v}}}(A)[W_{\mathrm{HF}}, \tau^s_{H_{\boldsymbol{v}}}(B)]\rangle_+ \,\mathrm{d}s + \mathcal{O}(\xi^2).$$

Since the dynamical system $(\mathrm{CAR}(\mathfrak{h}), \tau_{H_{\boldsymbol{v}}}, \langle\,\cdot\,\rangle_+)$ is quasi-free and $\langle A\,\mathrm{i}[W,B]\rangle_+ = \overline{\langle A^*\,\mathrm{i}[W,B^*]\rangle_+}$ holds for any $A, B, W \in \mathrm{CAR}(\mathfrak{h})$ with $W = W^*$, the proof of the estimate (3.9) reduces to showing that

$$\langle [W - W_{\mathrm{HF}}, A]B\rangle_\varrho = 0, \tag{4.21}$$

holds with $A = a(f)$, $B = a^*(g)$ and $A = a^*(f)$, $B = a(g)$ for any $f, g \in \mathfrak{h}$ and any density operator ϱ. Now a simple calculation yields

$$\begin{aligned}\langle [W, a^*(f)]a(g)\rangle_\varrho &= \langle g|\varrho v_{\mathrm{HF}} f\rangle = \langle [W_{\mathrm{HF}}, a^*(f)]a(g)\rangle_\varrho \\ \langle [W, a(g)]a^*(f)\rangle_\varrho &= -\langle g|v_{\mathrm{HF}}(I-\varrho)f\rangle = \langle [W_{\mathrm{HF}}, a(g)]a^*(f)\rangle_\varrho,\end{aligned}$$

thus establishing the validity of Relation (4.21). The local uniformity of the error term in Equ. (3.9) easily follows from the estimate (1.5) in [JOP3].

5 Conclusions and open problems

To the best of our knowledge, we provide for the first time sufficient conditions ensuring the existence of a steady state regime for the Green-Keldysh correlation functions of interacting fermions in mesoscopic systems in the partitioning and partition-free scenarios. Our proof handles these two cases in a unified way and even allows for mixed, thermodynamical and mechanical drive as well as adiabatic switching of these drives. We also show that the steady state, when it exists, is largely insensitive of the initial state of the system, depending only on its gross thermodynamical properties and not on structural properties like being a product state or a quasi-free state.

Roughly speaking, the most important technical conditions which insure the existence of a steady-state are two: *(i)* the non-interacting but fully coupled model has no bound states, and *(ii)* the strength of the self-interaction is small enough.

Under these conditions we do not have to perform an ergodic limit. As a practical application, we have shown that, up to second order corrections in the interaction strength, steady charge and energy currents coincide with their Hartree-Fock approximation which can be expressed in terms of a Landauer-Büttiker formula.

Let us point out some future steps towards a complete mathematical formulation of interacting quantum transport. Perhaps the most important progress would be to extend our scattering formalism to systems with bound states (i.e. weakly coupled quantum dots in the Coulomb blockade regime) and to a give a rigorous account on the diagrammatic recipes for the Green-Keldysh functions and interaction self-energies.

Another challenging issue is the existence of NESS for strongly correlated systems. This regime leads to the well known mesoscopic Kondo effect which relies on the Coulomb interaction between localized spins on the dot and the incident electrons from the leads. The theoretical treatment of this effect is notoriously difficult as the underlying Kondo and Anderson Hamiltonians do not allow perturbative calculations with respect to the interaction strength (see, e.g., the reviews [PG, H] and [KAO]).

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